Unit Root Tests for Time Series in the Presence of an Explosive Root

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WORKING PAPER 27/2008

February 2008

Price : Rs. 35

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K.Suresh Chandra and J.V.Janhavi

Abstract

This paper describes a modification for the practical relevance of unit root tests for time series generated by linear stochastic difference equations with an explosive root.

1. Introduction

The utility of unit roots tests in the process of transforming a non stationary time series into stationary time series, especially in the ARIMA modeling is well known in the literature. In particular, the Augmented Dickey Fuller (ADF) test is a valuable tool whenever the time series is generated by stochastic difference equation with a couple of suspected unit roots along with the roots that are stationary (or non-explosive) roots. ADF test is essentially a test with one-sided alternative, in that the rejection of the presence of a unit root by the ADF test leads to the conclusion that the stochastic difference equation is auto-regressive in nature. The main objective of this study is to expose the invalidity of the ADF test when the stochastic difference equation generating the given time series has an explosive root in addition to suspected unit roots and stationary roots. Based on a result due to Suresh Chandra, Manjunath and Vaman (1999), a modification is suggested that can enhance the practical utility of ADF tests when the time series has an explosive root.

With reference to the time series $\{Y_t,\ t=2,3,\ldots\}$ generated by any of the linear models

$$Y_t = \alpha + \rho Y_{t-1} + \varepsilon_t$$
$$Y_t = \alpha + \beta[t-(n-1)/2] + \rho Y_{t-1} + \varepsilon_t; \qquad t=2,3,\ldots \tag{1.1}$$

(Y_1 is fixed), where, $\{\varepsilon_t\}$ is a sequence of i.i.d. N(0, σ^2) random variables, Dickey and Fuller (1981) have investigated the likelihood ratio criterion to test H: $\rho=1$ against H_1: $|\rho|<1$. They have also derived the limiting distribution of the test statistic under both specifications in (1.1). Their investigations extend their results to general autoregressive models, for which the asymptotic equivalence of the limiting distribution of the least squares estimator of $(\alpha, \beta \text{ and } \rho)$ has been established.

To be specific, the generalization covers the specification (in their notation)

$$Y_t = \alpha + \rho Y_{t-1} + Z_t$$

or

$$Y_t = \alpha + \rho Y_{t-1} + \beta\left[t - \frac{n-p+1}{2}\right] + Z_t \tag{1.2}$$

where,

$$Z_t = Y_t - Y_{t-1}$$

and is generated by a stationary auto regressive process of order p (AR(p)) model

$$Z_t = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \ldots + \theta_p Z_{t-p} + \varepsilon_t \tag{1.3}$$

wherein $\{\varepsilon_t\}$ is a sequence of independent and identically distributed as N(0, σ^2) random variables. In view of the facts (Dickey and Fuller, 1981, p. 1066)

$$n^{-1}\sum_{t=2}^{n} Z_t = O_p(n^{-1/2})$$

$$\sum_{t=2}^{n} Y_{t-1}^2 = O_p(n^2)$$

$$n^{-2}\sum_{t=2}^{n}\left[t - \frac{n-p+1}{2}\right] Z_{t+p-j} = O_p(n^{-1/2})$$

$$\sum_{t=2}^{n} Y_{t-1} Z_{t-1} = O_p(n) \tag{1.4}$$

where, here and hence forth, $O_p(h_n)$ denotes, generically, terms which are bounded in probability, on being divided by h_n. Dickey and Fuller have established the asymptotic distributional equivalence of the least squares estimator of (α, β, ρ) in the models in (1.1) and (1.2) respectively, under H_A: α = 0, β = 0, ρ = 1.

2. Invalidity of Dickey – Fuller Test Under Explosive Conditions

The first objective in this paper is to demonstrate the invalidity of the asymptotic distributional equivalence of the least squares estimate of (α, β, ρ) in (1.1) and (1.2) respectively, under H_A, when the stochastic difference equation generating Z_t has an explosive root.

Towards elaborating this point, and to maintain consistency of notation, let us consider the model

$$Y_t^* = \alpha + \beta\left[t - \frac{n-p+1}{2}\right] + \rho Y_{t-1}^* + Z_t^* \tag{2.1}$$

where,

$$Y_t^* = Y_t - \tau Y_{t-1},\ \tau > 1.$$

Further,

$$\begin{aligned} Z_t^* &= Y_t^* - Y_{t-1}^* \\ &= (Y_t - \tau Y_{t-1}) - (Y_{t-1} - \tau Y_{t-2}) \\ &= Y_t - (1+\tau) Y_{t-1} + \tau Y_{t-2} \end{aligned} \tag{2.2}$$

and $\{Z_t^*\}$ is generated by the stationary autoregressive model

$$Z_t^* = \theta_1 Z_{t-1}^* + \ldots + \theta_p Z_{t-p}^* + \varepsilon_t \tag{2.3}$$

On setting $\{\varepsilon_t\}$ to be the usual i.i.d. N(0,σ^2) sequence, the model (2.1) is equivalent to

$$Y_t = \alpha + \beta\left[t - \frac{n-p+1}{2}\right] + \rho Y_{t-1} + Z_t \tag{2.4}$$

where, $\{Z_t\}$ is now generated by a stochastic difference equation of order (p+1):

$$Z_t = \eta_1 Z_{t-1} + \ldots + \eta_{p+1} Z_{t-p-1} + \varepsilon_t \tag{2.5}$$

When one of the roots, namely τ, of the polynomial equation

$$P(z) = z^{p+1} - \eta_1 z^p - \eta_2 z^{p-1} - \ldots - \eta_{p+1} = 0 \tag{2.6}$$

is larger than one.[1]

Hence $\{Z_t\}$ as a partially explosive non-stationary series (in deviance to the stationarity assumptions made in Dickey and Fuller (1981).

On invoking the results in Venkataraman (1968), one can easily verify that

[1] Most authors use the roots of polynomial equation $P^*(z) = 1 - \eta_1 z - \eta_2 z^2 - \ldots - \eta_{p+1} z^{p+1} = 0$ instead of those of P(z)=0 to classify the time series as stationary or non-stationary. The roots of $P^*(z)=0$ are the reciprocals of the roots of P(z)=0. To be precise, the roots of $P^*(z)=0$ that are numerically larger (smaller) than 1 are precisely the roots of P(z)=0 that are numerically smaller (larger) than 1. This ought to clarify any confusion that might arise when we refer to stationarity or otherwise with reference to the roots of P(z)=0 in relation to unity.

$$\sum_{t=2}^{n} Z_t = O_p(\tau^n)$$

$$\sum_{t=2}^{n} Y_{t-1}^2 = O_p(\tau^{2n})$$

$$\sum_{t=2}^{n} Y_{t-1} Z_{t-1} = O_p(\tau^{2n}) \tag{2.7}$$

$$\sum_{t=p-j+2}^{t=2} \left[t - \frac{n+p+1}{2} \right] Z_{t+p-j} = O_p(\tau^n)$$

The different rates of convergence of series in (1.4) and (2.7) are sufficient to reveal the invalidity of asymptotic equivalence of the least square estimator of (α, β, ρ), under H_A, relating to the equations (1.2) and (2.1).

This fact can also be conceived either by direct evaluation of plim $M_n^{-1} H_n M_n^{-1}$ (vide Dickey and Fuller, 1981, p. 1066), or, by setting $\hat{\rho}$ as the least squares estimate of ρ, and on noting from Venkataraman (1968) that, in the presence of the explosive root τ, $\left(\sum_{t=2}^{n} Y_{t-1}^2 \right)^{1/2} (\hat{\rho} - 1)$ converges in distribution to a random variable which can be expressed as a ratio of certain linear combinations of ε_t, which is not equivalent to the limiting random variable to which $\left(\sum_{t=2}^{n} Y_{t-1}^2 \right)^{1/2} (\hat{\rho} - 1)$ converges when τ is not present in the model (2.1) (vide Dickey and Fuller, 1981, p. 1060).

3. Modification Suggested When τ is Known

It is easy to note that the models (1.1) and (2.1) are equivalent when Y_t is transformed to $Y_t^* = Y_t - \tau Y_{t-1}$. (and consequently Z_t to Z_t^*). Hence the convergence rates of the series in (1.4) hold as such when Y_t and Z_t are replaced by Y_t^* and Z_t^* respectively. This leads easily to the asymptotic validity of Dickey and Fuller test for the transformed process $\{Y_t^*\}$ for testing for the unit root in (2.1).

4. Modification Suggested When τ is Unknown

When τ is unknown, but known to be larger than unity, we propose the estimator for τ given by

$$\hat{\tau} = \frac{\sum_{t=2}^{n} Y_t Y_{t-1}}{\sum_{t=2}^{n} Y_{t-1}^2} \tag{4.1}$$

It has been proved (Suresh Chandra, Manjunath and Vaman, 1999) that $\{ \tau^n (\hat{\tau} - \tau) \}$ is bounded in probability. Motivated by this result we suggest the transformation

$$\begin{aligned} \hat{Y}_t^* &= Y_t - \hat{\tau} Y_{t-1} \\ &= Y_t^* + (\tau - \hat{\tau}) Y_{t-1} \\ &= Y_t^* + g_1(t) \qquad (say) \end{aligned}$$

and hence

$$
\begin{aligned}
\hat{Z}_t^* &= \hat{Y}_t^* - \hat{Y}_{t-1}^* \\
&= (Y_t - \hat{\tau} Y_{t-1}) - (Y_{t-1} - \hat{\tau} Y_{t-2}) \\
&= (Y_t^* + (\tau - \hat{\tau}) Y_{t-1}) - (Y_{t-1}^* + (\tau - \hat{\tau}) Y_{t-2}) \\
&= Z_t^* + (\tau - \hat{\tau})(Y_{t-1} - Y_{t-2}) \\
&= Z_t^* + (\tau - \hat{\tau}) Z_{t-1} \\
&= Z_t^* + g_2(t) \qquad (say)
\end{aligned} \tag{4.2}
$$

Using (4.2), a substitutional evaluation and an algebraic simplification would facilitate the rewriting of (2.1) in the form

$$
\hat{Y}_t^* - g_1(t) = \alpha + \beta\left(t - \frac{n-p+1}{2}\right) + \rho\left(\hat{Y}_{t-1}^* - g_1(t-1)\right) + (\hat{Z}_t^* - g_2(t)) \tag{4.3}
$$

where,

$$
\begin{aligned}
&(1)\ g_2(t) = \left\{\sum_{i=1}^{p} \theta_i Z_{t-i}\right\}(\tau - \hat{\tau}) \\
&(2)\ \hat{Z}_t^* = \sum \theta_i \hat{Z}_{t-i}^* + \varepsilon_t
\end{aligned} \tag{4.4}
$$

where in the process $\{\hat{Z}_t^*\}$ given $\hat{\tau}$ is a stationary autoregressive process. In view of the boundedness in probability of $\{\tau^n (\hat{\tau} - \tau)\}$ it follows that each of

$$
\left\{(\tau - \hat{\tau}) \sum_{t=2}^{n} Y_t\right\}, \left\{(\tau - \hat{\tau}) \sum_{t=2}^{n} Z_t\right\}, \left\{(\tau - \hat{\tau}) \sum_{t=2}^{n}\left(t - \frac{n-p+1}{2}\right) Z_t\right\} \tag{4.5}
$$

is bounded in probability. This information, together with (4.3) and (4.4) would ultimately help us to show that

$$
\begin{aligned}
& n^{-1} \sum_{t=2}^{p} \hat{Z}_t^* = O_P\left(n^{-1/2}\right) \\
& \sum_{t=2}^{n} \left(\hat{Y}_t^*\right)^2 = O_P\left(n^2\right) \\
& n^{-2}\left(\sum_{t=2}^{n}\left(t - \frac{n-p+1}{2}\right) \hat{Z}_{t+p-j}^*\right) = O_P\left(n^{-1/2}\right) \\
& \sum_{t=2}^{n} \hat{Y}_{t-1}^* \hat{Z}_{t-1}^* = O_P(n)
\end{aligned} \tag{4.6}
$$

on appealing to standard convergence theorems.

These lead, on closely following the arguments in (Dickey and Fuller, 1981, pp. 1065 and 1066), with reference to the process $\{\hat{Y}_t^*\}$, that the least square estimator of (α, β, ρ) in (4.3) are asymptotically distributionally equivalent to those of the equation (2.1).

5. Some Remarks on the Practical Utility of the Proposed Modification

A crucial assumption we have made is that one of the roots of the model generating the time series is explosive, in the sense that it is numerically larger than unity. Its justification, from practical point of view can pose a methodological issue, especially when there is no standard statistical test for testing the existence of an explosive root. Constructing such tests based on $\hat{\tau}$, may pose theoretical problems in view of different rates of convergence and types of limits in distribution of $\hat{\tau}$ when τ is non-explosive or explosive or an unit root. However the numerical value of

$$\hat{\tau} = \frac{\sum_{t=2}^{n} Y_t Y_{t-1}}{\sum_{t=2}^{n} Y_{t-1}^2} \tag{5.1}$$

can be used for suspecting the presence of an explosive root. If, $\hat{\tau} \leq 1$ the ADF test appears to be consistent with the null and alternative that goes with it. However if $\hat{\tau} > 1$, the alternative appears to be logically not correct in view of the possibility of τ being larger than unity. The exponential rate of convergence in probability of $\hat{\tau}$ to τ, suggests that, τ is more likely to be larger than unity. Hence one can use it for identifying the presence of an explosive root, even in moderately large samples, although in small samples, there is a possibility of the effect of an explosive root being mimicked by a polynomial trend which gets eliminated by successive differences eventually.

It is pertinent to note that if $\hat{\tau} > 1$, any unit root test, particularly the Augmented Dickey-Fuller (ADF) test, is more likely to accept the hypothesis on the unit root, in which case, using $\hat{Y}_t = Y_t - \hat{\tau} Y_{t-1}$ instead of differencing can hasten the process of converting a non-stationary time series into a stationary time series, as seen from the real example that follows. It may be noted that $\hat{Y}_t = Y_t - \hat{\tau} Y_{t-1}$ theoretically eliminates the explosive root without affecting the presence or absence of the unit

root. Hence the suggested modification is useful even when there is no unit root in P(z)=0, In case there are unit roots in P(z)=0 along with an explosive root, one can apply the ADF test after removing the explosive root so that the alternative hypothesis in the ADF test is then logically correct when the test rejects the null hypothesis. This, we believe, enhances the practical utility of the ADF test in transforming the non-stationary time series into a stationary one, effectively.

6. An Illustrative Example

Towards illustrating the utility of the discussions so far, let us consider the data on Indian Exports from 1970-71 to 2003-04 as reported in Handbook of Statistics in Indian economy, RBI 2003-04. The increasing nature of the data suggests non-stationarity of the time series. One can easily note that ADF test accepts the unit root hypothesis as seen in the following results summary on using MICROFIT. In the table ADF(n) is the Augmented Dickey Fuller test with n difference components in the model. The null hypothesis gets rejected whenever the test statistic is smaller than the given critical values, at 5 percent level of significance.

TEST	**Statistic when there is no trend**	**Statistic when there is a linear trend**
ADF(1)	6.2012	3.1187
ADF(2)	3.8964	2.9900
ADF(3)	2.4874	2.0856
ADF(4)	2.3628	2.3706
Critical Value	-2.9750	-3.5867

Since the unit root hypothesis gets accepted, the results for the once differenced series are given below:

TEST	Statistic when there is no trend	Statistic when there is a linear trend
ADF(1)	0.3000	-2.2023
ADF(2)	1.0920	-1.2490
ADF(3)	1.2729	-1.0203
ADF(4)	2.0356	-0.2837
Critical Value	-2.9750	-3.5867

Even at this stage the unit root hypothesis gets accepted and it requires one more differencing to make the series stationary, at least for the model with one lagged difference, as revealed by the following table for the twice differenced series.

TEST	Statistic when there is no trend	Statistic when there is a linear trend
ADF(1)	-4.1277	-4.0416
ADF(2)	-2.7642	-2.7052
ADF(3)	-2.4938	-2.4425
ADF(4)	-1.7315	-1.6832
Critical Value	-2.9750	-3.5867

However, using the formula in (5.1) we get $\hat{\tau} = 1.153104$ which is numerically larger than unity suggesting the explosive nature of the time series. Eliminating this root using the formula $\hat{Y}_t = Y_t - \hat{\tau}\, Y_{t-1}$ and applying the ADF test for the new series we have the following summary.

TEST	Statistic when there is no trend	Statistic when there is a linear trend
ADF(1)	-4.0971	-4.0019
ADF(2)	-2.7366	-2.6676
ADF(3)	-2.4548	-2.3966
ADF(4)	-1.6965	-1.6436
Critical Value	-2.9750	-3.5867

The above table reveals that the explosive root eliminated series is stationary, at least for the model under ADF(1) test. In fact, the result indicates that there are no unit roots and suggests an AREXMA model - ARMA model with an explosive root and with no unit roots - for the time series.

7. In Conclusion

It has been proved (Suresh Chandra, Manjunath and Vaman, 1999) that $\hat{\tau}$ in (5.1) consistently estimates the largest explosive root τ, of P(z)=0. Consequently, whenever $\hat{\tau}$ >1 one can expect, in large samples, the modified series $\hat{Y}_t = Y_t - \hat{\tau}\, Y_{t-1}$, to eliminate the largest explosive root (τ). One can easily extend the suggested procedure to eliminate, successively, more than one explosive roots (distinct or multiple), until the estimate (5.1), based on such successively modified series is less than unity. In fact, all such estimates will have exponential rates for their convergence in probability (vide Suresh Chandra *et al*, 1999) as long as there is an explosive root.

Finally, the elimination of explosive roots before applying the unit root tests not only hastens the process of removing non-stationarity from a practical point of view (as seen in the illustration above), but also validates the one sided assumption of the alternative hypothesis in them. It is from this perspective the discussions in this paper gains its importance in empirical time series analysis. The possibility of an alternative to ARIMA modeling, as revealed in the illustration given above, can also be exploited.

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