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**Stochastic Dominance and Mean-Variance
Measures of Profit and Loss for Business
Planning and Investment**

by

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Abstract

In this paper, we first extend the stochastic dominance (SD) theory by introducing the first three orders of both ascending SD (ASD) and descending SD (DSD) to decisions in business planning and investment to risk-averse and risk-loving decision makers so that they can compare both return and loss. We provide investors with more tools for empirical analysis, with which they can identify the first order ASD and DSD prospects and discern arbitrage opportunities that could increase his/her utility as well as wealth and set up a zero dollar portfolio to make huge profit. Our tools also enable investors and business planners to identify the third order ASD and DSD prospects and make better choices.

To complement the stochastic dominance approach, we also introduce an improved mean-variance criterion to decisions in business planning or investment on both return and loss for risk-averse and risk-loving investors. We then illustrate the superiority of the present approaches with well-known examples in the literature and discuss the relationship between the improved stochastic dominance and mean-variance criteria.

1. Introduction

Since the development of the classical mean-variance (MV) context by Markowitz (1952) and Tobin (1958), the behavior towards risk of both risk averters and risk lovers have been widely studied by applying the MV approach which measures the risk exposure of financial assets and portfolios of financial assets. Another measure, stochastic dominance (SD), as partial orders defined over a set of risky payoffs, also provides useful criterion for portfolio choice and risk management. Originated from the majorization theory (Hardy et al, 1934; Marshall and Olkin, 1979), the stochastic dominance is formally developed by Quirk and Saposnik (1962), Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970). The SD approach has been regarded as one of the most useful tools to rank investment prospects when there are uncertainties (see, for example, Levy 1992) as the ranking of the assets has been proven to be equivalent to utility maximization for the preferences of risk averters and risk lovers (see, for example, Quirk and Saposnik, 1962; Hanoch and Levy, 1969; Hammond, 1974; Stoyan, 1983; Li and Wong, 1999).

Since SD rules have been demonstrated to offer, in many cases, superior and more efficient criteria on which to base investment decisions than the criterion derived from the traditional model of asset choice based on MV methodology, the use of SD theory to compare profit or return for risk averters has been well established in both theory and application. Theoretical works linking the SD theory to the selection rules for risk averters and risk lovers under different restrictions on the utility functions has also been well investigated (for example, Quirk and Saposnik 1962 and Hammond 1974). To distinguish the SD theory for risk averters from that for risk lovers, in this paper, we call the former ascending stochastic dominance (ASD) and the latter descending stochastic dominance (DSD). The SD theory to compare profit or return for risk averters and risk lovers has also been widely applied in business and economics. For example, Hershey and Schoemaker (1980) investigate risk loving in the domain of losses, and Kuosmanen

(2004) derives some simple SD efficiency measures and test statistics to analyze industrial diversification of the market portfolio.

The concept of SD and utility function theory has also been used to compare risk profiles or loss of various investments. Hershey and Schoemaker (1980) present an experimental investigation of risk taking and show that the results are partially compatible with expected utility analysis. Fennema and van Assen (1998) introduce a measure of the utility of losses by means of the tradeoff method. Post and Diltz (1986), Weeks and Wingler (1979) and Weeks (WKS, 1985) discuss the behaviour of the risk-averse decision maker when the potential loss of a project is used as the variable of interest. Dillinger et al. (DSM, 1992) identify the mistakes in Post and Diltz (1986), Weeks, and Wingler (1979) and WKS with corrected modification by introducing the Variability Ordering, which is a dual problem of the second order ASD in our paper.

Working along similar lines as Whitmore (1970) who extends the second order SD developed by Quirk and Saposnik (1962) and others to the third order SD for risk averters, in this paper we first extend the works of Hershey and Schoemaker (1980), Post and Diltz (1986), Weeks and Wingler (1979), WKS and DSM by introducing the first three orders ASD to study the behavior of risk averters on the potential loss as well as the profit of a project. As risk-loving behaviour is an important issue (see, for example, Hammond, 1974; Hershey and Schoemaker, 1980; Stoyan, 1983; Myagkov and Plott, 1997; Anderson, 2004; Post and Levy, 2005), we then extend the works of WKS and DSM and others by introducing the first three orders of DSD to decisions in business planning and investment for risk-averse as well as risk-loving decision makers to compare both return and loss. Our approaches take into consideration the entire distributions of the loss and profit of different business opportunities and compares both the downside risk and upside profit opportunities. Though the theory can be easily extended to any order, in this paper we focus our discussion on the first three as the first three orders SD are of most importance in theory as well as in empirical application. The first order SD is very important as it gives rise to arbitrage opportunities (Bawa, 1978; Jarrow, 1986). The second order SD is prominent in SD literature as it can be used by

risk averters and risk lovers to choose among different investment opportunities. On the other hand, many academics demonstrate the usefulness of the third order SD, see for example, Whitmore (1970), Whitmore and Findley (1978), Shorrocks and Foster (1987), Wong and Li (1999), Gotoh and Konno (2000), Ng (2000) and Anderson (2004).

As a complement of the SD theory, there are the MV or mean-standard deviation (MS) preferences of the decision-maker to the variable of profit derived from a von Neumann-Morgenstern (von Neumann and Morgenstern, NM, 1944) quadratic utility function and from a family of normal distributions. They are often used in economics and business, especially after Markowitz (1952) and Tobin (1958) propose the MV selection rules for risk-averters. For example, Konno and Yamazaki (1991) apply the MV preference to optimize the stock portfolio with illustrations from the Tokyo Stock Market while McNamara (1998) and Gasbarro et al. (2006) apply both MV and SD to exploit the association of returns on risky assets. On the other hand, WKS apply the theory of the MV criterion to the variable of loss. In this paper, we extend the work of Markowitz, Tobin and WKS by introducing the improved MV criterion on both the variable of profit and the variable of loss to decision making in business planning and investment for risk-averse and risk-loving investors.

The new SD and MV theoretically-driven methodologies developed in our paper enables business planners and investors to analyse many complex contemporary decision problems that could occur in various forms including inter-organizational, group-based, and technology-enabled. We examine the possibilities for the practical implementation used by the financial decision maker and emphasize the impact of the application made by the theories developed in our paper. To do that, in this paper we illustrate the superiority of our approaches to the analysis of both SD and MV with the examples used in WKS, DSM and Levy and Levy (2002). In our illustration, the methodologies developed in our paper can identify the first order ASD and DSD prospects and discern arbitrage opportunities that could increase one's utility as well as wealth and set up a zero dollar portfolio to make huge profit. Our tools also enable investors and business planners

to identify the third order ASD and DSD prospects and make better choices. We further explore the relationship between SD and MV with illustrations.

2. Definitions and Notations

Let $\overline{\mathbf{R}}$ be the set of extended real numbers and $\Omega = [a,b]$ be a subset of $\overline{\mathbf{R}}$ in which a and b can be finite or infinite. Let $\mathbf{B}$ be the Borel σ-field of Ω and μ be a *measure* on $(\Omega, \mathbf{B})$. The functions F and F^D of the measure μ are defined as:

$$F(x) = \mu\,[a,\,x] \quad \text{and} \quad F^D(x) = \mu\,[x,\,b] \text{ for all } x \in \Omega\,. \tag{1}$$

The function F is called a *probability distribution function* or *cumulative distribution function* (CDF) and μ is called a *probability measure* if $\mu(\Omega) = 1$. By the basic probability theory (Ash, 1972), for any random variable X and for any probability measure P, there exist a unique induced probability measure μ on $(\Omega, \mathbf{B})$ and a probability distribution function F which satisfies (1) and

$$\mu(B) = P(X^{-1}(B)) = P(X \in \mathrm{B}) \quad \text{for any} \quad B \in \mathbf{B}.$$

An integral written in the form of $\int_A f(t)d\mu(t)$ or $\int_A f(t)dF(t)$ is a Lebesgue integral for any integrable function $f(t)$. For any $A = [c,d] \subseteq \Omega$, we will write the integral as $\int_c^d f(t)dt$. We note that as Lebesgue integral measures any integration of any continuous random variable as well as any summation of any discrete random variable, in this paper we will use $\int_c^d f(t)dt$ to present an integration of any continuous random variable from c to d as well as any summation of any discrete random variable from c to d for any $[c,d] \subseteq \Omega$.

Variables denoted by X, Y, ⋯ defined on Ω are considered random variables. The cumulative distribution functions of X and Y are F and G respectively and let f and g be

the corresponding probability density functions (PDFs) of two prospects, *X* and *Y*, respectively, with common support of $\Omega = [a, b]$, where $a < b$, then the following notations will be used throughout this paper:

$$\mu_F = \mu_X = E(X) = \int_a^b x dF(x), \quad \mu_G = \mu_Y = E(Y) = \int_a^b x dG(x),$$

$$M_0^A = M_0^D = m, \ M_j^A(x) = \int_a^x M_{j-1}^A(t)dt \text{ and } M_j^D(x) = \int_x^b M_{j-1}^D(t)dt \qquad (2)$$

for $m = f, g$; $M = F, G$; and $j = 1, 2, 3$. The definition of the first, second and third order ascending stochastic dominances (ASD) applied to risk averters is then defined as follows:

Definition 1: Given two random variables X and Y with F and G as their respective cumulative distribution functions defined on [a, b], X is at least as large as Y and F is at least as large as G in the sense of:

    a. *FASD, denoted by $X \succeq_1 Y$ or $F \succeq_1 G$, if and only if $F_1^A(x) \leq G_1^A(x)$ for all x in [a, b],*

    b. *SASD, denoted by $X \succeq_2 Y$ or $F \succeq_2 G$, if and only if $F_2^A(x) \leq G_2^A(x)$ for all x in [a, b], and*

    c. *TASD, denoted by $X \succeq_3 Y$ or $F \succeq_3 G$, if and only if $F_3^A(x) \leq G_3^A(x)$ for all x in [a, b] and $\mu_F \geq \mu_G$,*

        where FASD, SASD and TASD stand for the first, second and third order ascending stochastic dominances respectively, and μ_F and μ_G are the means of X and Y respectively.

If, in addition, there exists x in [a, b] such that $F_i^A(x) < G_i^A(x)$ for i = 1, 2 and 3, we say that X is larger than Y and F is larger than G in the sense of SFASD, SSASD and STASD, denoted by $X \succ_1 Y$ or $F \succ_1 G$, $X \succ_2 Y$ or $F \succ_2 G$ and $X \succ_3 Y$ or $F \succ_3 G$ respectively, where SFASD, SSASD, and STASD stand for strictly first, second and third order ascending stochastic dominances respectively.

The definition of the first, second and third order descending stochastic dominances (DSD) applied to risk lovers is defined as follows:

Definition 2 : Given two random variables X and Y with F and G as their respective cumulative distribution functions defined on [a, b], X is at least as large as Y and F is at least as large as G in the sense of:

  a. *FDSD, denoted by $X \succeq^1 Y$ or $F \succeq^1 G$, if and only if $F_1^D(x) \geq G_1^D(x)$ for all x in [a, b],*

  b. *SDSD, denoted by $X \succeq^2 Y$ or $F \succeq^2 G$, if and only if $F_2^D(x) \geq G_2^D(x)$ for all x in [a, b], and*

  c. *TDSD, denoted by $X \succeq^3 Y$ or $F \succeq^3 G$, if and only if $F_3^D(x) \geq G_3^D(x)$ for all x in [a, b] and $\mu_F \geq \mu_G$,*

     where FDSD, SDSD and TDSD stand for the first, second and third order descending stochastic dominances respectively, and μ_F and μ_G are the means of X and Y respectively.

If, in addition, there exists x in [a, b] such that $F_i^D(x) > G_i^D(x)$ for i = 1, 2 and 3, we say that X is larger than Y and F is larger than G in the sense of SFASD, SSASD and STASD, denoted by $X \succ^1 Y$ or $F \succ^1 G$, $X \succ^2 Y$ or $F \succ^2 G$ and $X \succ^3 Y$ or $F \succ^3 G$ respectively, where SFDSD, SSDSD, and STDSD stand for strictly first, second and third order descending stochastic dominances respectively.

We call the integrals M_n^A defined in (2) to be FASD, SASD and TASD integrals and M_n^D defined in (2) to be FDSD, SDSD and TDSD integrals respectively for $n = 1$, 2 and 3 and for $M = F$ and G. Let X be the variable of return or profit with the cumulative distribution function F and u be the utility function associating with profit. Following WKS and DSM, we define the variable of loss X^* with the corresponding cumulative distribution function F^* and the utility function u^* associating with loss in the following definition:

Definition 3: Suppose that X is the variable of return or profit with the cumulative distribution function F defined on [a, b], the variable of loss X^* with the corresponding cumulative distribution function F^* defined on [-b, -a] is:

$$X^* = -X \quad . \tag{3}$$

Suppose u is the utility-for-profit function associating with the variable of profit and u^* is the corresponding utility-of-loss function associating with the variable of loss, then

$$u^*(X^*) = u(X) \quad . \tag{4}$$

We note that the variable of loss X^* in (3) can also be defined as

$$X^* = TC - X \tag{5}$$

where *TC* is the total cost (assumed to be fixed) and *X* is the profit. However, both definitions of (3) and (5) will draw the same conclusions based on both SD and MV criteria. This argument can be proven by modifying the proofs in Theorems 4 in Hadar and Russell (1971) and Theorem 1' in Tesfatsion (1976). In this connection, we will use (3) as well as (5) in the entire paper.

We note that Definitions 1 and 2 can be used to compare both profit and loss in a way that both *X* and *Y* can be the variable of profit or the variable of loss. Since we denote *X** (*Y**) as the variable of loss with the corresponding cumulative distribution function *F** (*G**), we refine ASD and DSD for the variable of loss as follows:

Definition 4: Given two random variables of loss X* and Y* with F* and G* as their respective cumulative distribution functions defined on [-b, -a], X* is at least as large as Y* and F* is at least as large as G* in the sense of:

a. *FASD, denoted by $X^* \succeq_1 Y^*$ or $F^* \succeq_1 G^*$, if and only if $F^{*A}_1(x^*) \leq G^{*A}_1(x^*)$ for all x in [-b, -a],*

b. *SASD, denoted by $X^* \succeq_2 Y^*$ or $F^* \succeq_2 G^*$, if and only if $F^{*A}_2(x^*) \leq G^{*A}_2(x^*)$ for all x in [-b, -a], and*

c. *TASD, denoted by $X^* \succeq_3 Y^*$ or $F^* \succeq_3 G^*$, if and only if $F^{*A}_3(x^*) \leq G^{*A}_3(x^*)$ for all x^* in [-b, -a] and $\mu_F^* \geq \mu_G^*$,*

where μ_F* and μ_G* are the means of $X*$ and $Y*$ respectively.

Definition 5: Given two random variables of loss $X*$ and $Y*$ with $F*$ and $G*$ as their respective cumulative distribution functions defined on $[-b, -a]$, $X*$ is at least as large as $Y*$ and $F*$ is at least as large as $G*$ in the sense of:

    a. *FDSD, denoted by $X* \succeq^1 Y*$ or $F* \succeq^1 G*$, if and only if $F*_1^D(x*) \geq G*_1^D(x*)$ for all $x*$ in $[-b, -a]$,*

    b. *SDSD, denoted by $X* \succeq^2 Y*$ or $F* \succeq^2 G*$, if and only if $F*_2^D(x*) \geq G*_2^D(x*)$ for all $x*$ in $[-b, -a]$, and*

    c. *TDSD, denoted by $X* \succeq^3 Y*$ or $F* \succeq^3 G*$, if and only if $F*_3^D(x*) \geq G*_3^D(x*)$ for all $x*$ in $[-b, -a]$ and $\mu_F* \geq \mu_G*$,*

where μ_F* and μ_G* are the means of $X*$ and $Y*$ respectively.

The definitions of the strictly ASD and DSD for the variables of loss can be defined similarly. In the basic theorems in SD theory, the ASD and DSD are used to compare profits by matching certain utility functions with the variable of profit or return as shown in the following definition:

Definition 6:   Let u be utility-for-profit function. For $n = 1, 2, 3$, U_n^A, U_n^{SA}, U_n^D and U_n^{SD} are sets of utility-for-profit functions such that:

$$U_n^A(U_n^{SA}) = \{u : (-1)^{i+1} u^{(i)} \geq (>)0, i = 1, \cdots, n\},$$

$$U_n^D(U_n^{SD}) = \{u : u^{(i)} \geq (>)0, i = 1, \cdots, n\}.$$

where $u^{(i)}$ is the ith derivative of the utility function u.

Note that the theory can be easily extended to satisfy utilities defined in Definition 6 to be non-differentiable. For simplicity, we skip the discussion of non-differentiable utilities in this paper. Let u^* be utility-of-loss function. The classes of utility-of-loss functions corresponding to the variables of loss are defined as follows:

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Definition 7: For $n = 1, 2, 3$, U_n^{*A}, U_n^{*SA}, U_n^{*D} and U_n^{*SD} are sets of utility-of-loss functions such that:

$$U_n^{*A}(U_n^{*SA}) = \{u^* : u^*(X^*) = u(X), \ u \in U_n^A(U_n^{SA})\} ; \ \text{and}$$

$$U_n^{*D}(U_n^{*SD}) = \{u^* : u^*(X^*) = u(X), \ u \in U_n^D(U_n^{SD})\} ;$$

where X^* is defined in (3) and u^* is defined in (4).

Weeks and Wingler (1979) find that the utility-of-loss function is the mirror image (around the vertical axis) of the same investor's utility-for-profit function. Hence, we define the set of utility-of-loss functions for the same investors whose utility-for-profit functions belong to U_n^A as U_n^{*A}. Similarly, other sets of utility-of-loss functions are defined in the same way. Note that in Definition 6 'increasing' means 'non-decreasing' and 'decreasing' means 'non-increasing'. We also remark in Definition 6 that $U_1^A \equiv U_1^D$ and $U_1^{SA} \equiv U_1^{SD}$. An individual chooses between F and G in accordance with a consistent set of preferences satisfying the NM consistency properties. Accordingly, F is (strictly) preferred to G, or equivalently, X is (strictly) preferred to Y, if

$$\Delta u = u(F) - u(G) \geq 0(> 0) \ \text{and} \ \Delta u^* = u^*(F^*) - u^*(G^*) \geq 0(> 0) \qquad (6)$$

where $u(F) = E[u(X)]$, $u(G) = E[u(Y)]$, $u^*(F^*) = E[u^*(X^*)]$, $u^*(G^*) = E[u^*(Y^*)]$.

The above definitions enable business planners or investors to use the entire distributions of the variables of loss and the variables of profit to compare different investment opportunities. To examine the downside risk of different orders, business planners could apply the ASD approach of the corresponding orders to the variables of profit (see Definition 1) or apply DSD of the corresponding orders to the variables of loss (see Definition 5). Our approaches also enable investors to take into consideration the upside profit of different orders in their decisions by applying DSD to the variables of profit of the corresponding orders (see Definition 2) and applying ASD to the variables of loss of the corresponding orders (see Definition 4). In addition, our approaches also make it possible for decision makers to review the effects of both downside risk and upside profit of different orders at the same time before making decisions about their investments, be they risk-averse or risk-loving decision makers.

3. Stochastic Dominance Criterion for Decision-Making in Business Planning

DSM introduce the concept of *Stochastically More Variable* (*SMV*) to study the variables of loss as well as return and suggest that a risk-averse decision maker will follow the ordering of random variables given by SASD while a risk-loving decision maker will follow the variability ordering instead. We note that DSM call SASD Second Order Stochastic Dominance (SSD) and the concept of SMV introduced by DSM is a dual problem of SASD and it is the same concept as SDSD in Definition 2. In order to distinguish loss from profit, in this paper we define clearly the variables of profit and variables of loss and their correspondence utility functions. In addition, to extend the theory introduced by WKS and DSM to the first three orders and to include both utility-for-profit and utility-of-loss functions, we employ the theory of both ASD and DSD so that risk-averse and risk-loving decision makers can make decisions in business planning and investment prospects.

Weeks and Wingler (1979) claim that the utility-of-loss functions are mirror images (around the vertical axis) of the same investor's utility-for-profit functions. In addition, it is easy to prove that (1) u^* is a decreasing function since u is an increasing function, (2) u^* is convex (concave) if and only if u is concave (convex), (3) the third derivative of u^* is of different sign from that of u, and (4) $U_1^A = U_1^D$ is the set of increasing functions of the utilities for profit while $U_1^{*A} = U_1^{*D}$ is the set of decreasing functions of the utilities of loss. The following theorem[1] and corollary provide the linkage between X and X^* in the SD theory:

Theorem 1 : For random variables X and Y, we have the following:

    a. $X \succeq_i (\succ_i) Y$ if and only if $Y^* \succeq^i (\succ^i) X^*$ for i =1, 2 or 3.

    b. $X \succeq_1 (\succ_1) Y$ if and only if $X \succeq^1 (\succ^1) Y$.

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[1] We skip reporting the proofs of all the theorems in the paper for simplicity. They are available on request.

c. If X and Y have the same finite mean, then

$$X \succeq_2 (\succ_2) Y \text{ if and only if } Y \succeq^2 (\succ^2) X \; .$$

Corollary 2:   For random variables of profit, X and Y, we have the following:

a. $X^* \succeq_i (\succ_i) Y^*$ if and only if $Y \succeq^i (\succ^i) X$ for i =1, 2 or 3.

b. $X^* \succeq_1 (\succ_1) Y^*$ if and only if $X^* \succeq^1 (\succ^1) Y^*$.

c. If X and Y have the same finite mean, then

$$X^* \succeq_2 (\succ_2) Y^* \text{ if and only if } Y^* \succeq^2 (\succ^2) X^*.$$

where X^* and Y^* are the variables of loss defined in (3).

The following theorem states the relationship between the SD preferences and the utility preferences for risk averters and risk lovers with respect to the utility functions of profit:

Theorem 3:   Let X and Y be random variables of profit with cumulative distribution functions F and G respectively. Suppose u is a utility-for-profit function, for $m = 1, 2$ and 3, we have the following:

a. $F \succeq_m (\succ_m) G$ if and only if $u(F) \geq (>) u(G)$ for any u in $U_m^A (U_m^{SA})$; and

b. $F \succeq^m (\succ^m) G$ if and only if $u(F) \geq (>) u(G)$ for any u in $U_m^D (U_m^{SD})$;

where $u(F)$ and $u(G)$ are defined in (6).

The following theorem states the relationship between the SD preferences and the utility preferences for risk averters and risk lovers with respect to the utility functions of loss:

Theorem 4: Let X^* and Y^* be random variables of loss with cumulative distribution functions F^* and G^*, and u^* be a utility-of-loss function satisfying (4). For $m = 1, 2$ and 3, we have the following:

a. $F^* \succeq_m (\succ_m) G^*$ if and only if $u^*(G^*) \geq (>) u^*(F^*)$ for any u^* in $U_m^{*D} (U_m^{*SD})$; and

b. $F^* \succeq^m (\succ^m) G^*$ if and only if $u^*(G^*) \geq (>) u^*(F^*)$ for any u^* in $U_m^{*A} (U_m^{*SA})$;

where $u^*(F^*)$ and $u^*(G^*)$ are defined in (6).

Theorems 3 and 4 show that the different classes of utility functions match the corresponding classes of stochastic dominance totally. These results are crucial in the stochastic dominance theory as SD can then be used for utility maximization which in turn provides tools for business planners, including risk-averse and risk-loving planners, to make their decisions in business planning and investment. For convenience, we call a person a first-order-ascending-stochastic-dominance (FASD) investor if his/her utility-for-profit function u belongs to U_1^A or if his/her utility-of-loss function belongs to U_1^{*A}; a first-order-descending-stochastic-dominance (FDSD) investor if his/her utility-for-profit function u belongs to U_1^D or if his/her utility-of-loss function belongs to U_1^{*D}; and a second-order-ascending-stochastic-dominance (SASD) risk investor if his/her utility-for-profit function u belongs to U_2^A or if his/her utility-of-loss function belongs to U_2^{*A}. A second-order-descending-stochastic-dominance (SDSD) risk investor, a third-order-ascending-stochastic-dominance (TASD) risk investor and a third-order-descending-stochastic-dominance (TDSD) risk lover investor can be defined in the same way. Since FASD and FDSD are equivalent (see Part b of Theorem 1), we will call the FASD and FDSD investors FSD investors. For simplicity, from now on we will use U_n^A (U_n^{*A}) to stand for both U_n^A (U_n^{*A}) and $U_n^{SA}(U_n^{*SA})$ and use U_n^D (U_n^{*D}) to stand for both U_n^D (U_n^{*D}) and $U_n^{SD}(U_n^{*SD})$. In the next section, we will discuss our extension of the MV criterion to both variables of profit and variables of loss as a complement to the SD approach, and then illustrate the superiority of our approaches in the illustration section.

4. Mean-Variance Criterion for Decision-Making in Business Planning

Applying the MV criterion to the variables of profit has been well established in the literature. WKS first applies the MV criterion to the variables of loss. In this paper, we extend the work further by improving the MV criterion so that it can be applied to the

variables of profit as well as the variables of loss and can be used by any risk-averse and risk-loving investors to make decisions in business planning and investment. For any two prospects with the variables of profit or return Y_i and Y_j with means μ_i and μ_j and standard deviations σ_i and σ_j respectively, it is well-known that Y_j is said to dominate Y_i by the MS rule if $\mu_j \geq \mu_i$ and $\sigma_j \leq \sigma_i$ (Markowitz, 1952; Tobin, 1958). This rule is standard in the literature for risk-averse investors. In this paper, we improve the MV criterion by introducing the ascending and descending MV rule to cover both risk averters and risk lovers as shown in the following two definitions:

Definition 8: Given two random variables of profit X and Y with means μ_x and μ_y and standard deviations σ_x and σ_y respectively, then

    a. X is said to dominate Y (strictly) by the Ascending MS (AMS) rule, denoted by $X \ MS_A \ Y$ if $\mu_x \geq (>)\mu_y$ and $\sigma_x \leq (<)\sigma_y$; and

    b. X is said to dominate Y (strictly) by the Descending MS (DMS) rule, denoted by $X \ MS_D \ Y$ if $\mu_x \geq (>)\mu_y$ and $\sigma_x \geq (>)\sigma_y$.

Definition 9: Given two random variables of loss X^* and Y^* with means μ_x^* and μ_y^* and standard deviations σ_x^* and σ_y^* respectively, then

    a. X^* is said to dominate Y^* (strictly) by the AMS rule, denoted by $X^* \ MS_A^* \ Y^*$ if $\mu_x^* \leq (<)\mu_y^*$ and $\sigma_x^* \leq (<)\sigma_y^*$; and

    b. X^* is said to dominate Y^* (strictly) by the DMS rule, denoted by $X^* \ MS_D^* \ Y^*$ if $\mu_x^* \leq (<)\mu_y^*$ and $\sigma_x^* \geq (>)\sigma_y^*$.

It is well-known that SASD is equivalent to MV efficiency when the variables are normally distributed (Markowitz, 1952; Tobin, 1958). Meyer (1987) extends the theory to include variables that differ only by location-scale parameters. We extend the efficiency further to cover a location-scale family and a combination of location-scale families. Based on Markowitz's and Meyer's findings, we then develop the linkage of the MS

rules to the utility maximization for both the variable of profit and the variable of loss, and relate it to the risk-averse and risk-loving investors in the following two theorems:

Theorem 5: Let X and Y be random variables of profit with means μ_x and μ_y and standard deviations σ_x and σ_y respectively.

    a. If X MS_A Y (strictly) and if both X and Y belong to the same location-scale family or the same linear combination of location-scale families, then $E[u(X)] \geq (>)E[u(Y)]$ for the risk-averse investor with the utility-for-profit function u in $U_2^A(U_2^{SA})$; and

    b. if X MS_D Y (strictly) and if both X and Y belong to the same location-scale family or the same linear combination of location-scale families, then $E[u(X)] \geq (>)E[u(Y)]$ for the risk-loving investor with the utility-for-profit function u in $U_2^D(U_2^{SD})$.

Theorem 6: Let X^* and Y^* be random variables of loss with means μ_x^* and μ_y^* and standard deviations σ_x^* and σ_y^* respectively.

    a. If X^* MS_A^* Y^* (strictly) and if both X^* and Y^* belong to the same location-scale family or the same linear combination of location-scale families, then $E[u^*(X^*)] \geq (>)E[u^*(Y^*)]$ for the risk-averse investor with the utility-of-loss function u^* in $U_2^{*A}(U_2^{*SA})$; and

    b. if X^* MS_D^* Y^* (strictly) and if both X^* and Y^* belong to the same location-scale family or the same linear combination of location-scale families, then $E[u^*(X^*)] \geq (>)E[u^*(Y^*)]$ for the risk-loving investor with the utility-of-loss function u^* in $U_2^{*D}(U_2^{*SD})$.

The above theorems will be useful in applying the improved MV criterion in the comparison of profit and loss. In addition, by incorporating the above two theorems into

Theorems 3 and 4, we provide linkage between the SD and MV rules. The application of these theorems will be illustrated in the next section.

5. Illustration

We adopt the examples used in WKS, DSM and Levy and Levy (2002) to illustrate the superiority of our approach. We first use the Production/Operations Management (POM) example demonstrated by WKS and DSM. A production/operations system needs extra capacity to satisfy the expected increased demand. Three mutually exclusive alternative sites have been identified and the cost (X^*) with their associated probabilities f^*, g^* and h^* have been estimated as shown in Table 1. The varying costs and accompanying probabilities reflect uncertainties in labor costs, shipping costs, construction costs, or other costs associated with the location alternatives. Table 1 also depicts the ASD integrals of the first three orders for each location.

Place Table 1 here

In the POM example, we extend the findings from WKS and DSM in the POM example for FASD, SASD, TASD risk averters as well as FDSD, SDSD, TDSD risk lovers to both variables of profit and variables of loss in detail. The example shows the risk profiles for three locations for a plant in terms of costs, see Table 1. The variable X can be defined as the variable of profit or return or as the variable of loss or cost. However, to avoid possible confusion, in this paper we use X^* to represent the variable of loss or cost and X to represent the variable of profit or return as defined in (3) or (5). In Tables 1 to 4, x^* represents different levels of costs with probabilities f^*, g^* and h^* and their corresponding CDFs F^*, G^* and H^* for the three different locations. The ASD and DSD integrals M_n^{*A} and M_n^{*D} defined in (2) for $n = 1$, 2 and 3 and $M = F$, G and H and the estimates of the ASD integrals are presented in Table 1 while the estimates of the DSD integrals for the three locations are depicted in Table 2.

Place Table 2 here

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Tables 3 and 4 show the differences of the ASD integrals and the DSD integrals for each pair of the three locations of the first three orders. To simplify the notations in Tables 3 and 4, we let the ASD Integral Differential PQ_n^{*A} and DSD Integral Differential PQ_n^{*D} be

$$PQ_n^{*A} = P_n^{*A} - Q_n^{*A} \text{ and } PQ_n^{*D} = P_n^{*D} - Q_n^{*D} \tag{7}$$

respectively, for $n = 1, 2$ and 3 and for $P, Q = F, G$ or H.

Place Tables 3 and 4 here

From Table 3, we find that neither F^*, G^* nor H^* dominates one another in the sense of FASD. However, from the table we find that $F^* \succ_n G^* \succ_n H^*$ for $n = 2$ and 3. From Theorem 4, we conclude that H^* is preferred to G^* which in turn is preferred to F^* by the SDSD or TDSD risk lover. Similarly, from Table 4, we find that neither F^*, G^* nor H^* dominates one another in the sense of FDSD. Table 4 also shows that F^* does not dominate G^* nor H^* in the sense of SDSD but $H^* \succ^2 G^*$ which implies that G^* is preferred to H^* by the SASD risk averter. Moreover, from the table, we find that $H^* \succ^3 G^* \succ^3 F^*$. Hence F^* is preferred to G^* which in turn is preferred to H^* by the TASD risk averter. This conclusion is different from DSM's as we draw inference to cover bigger classes of investors.

In addition, using the improved MV criterion, from Table 1, we have $G^* MS_A^* H^*$, and $H^* MS_D^* G^* MS_D^* F^*$. This means that G^* dominates H^* by the AMS rule while H^* dominates G^* which in turn dominates F^* by the DMS rule. From Theorem 6, we conclude that if G^* and H^* belong to the same location-scale family, then G^* is preferred to H^* by the SASD risk averter. Similarly, if H^*, G^* and F^* belong to the same location-scale family, then H^* is preferred to G^* which in turn is preferred to F^* by the SDSD risk lover. Similarly, our conclusions are different from both WKS' and DSM's as we draw inference to cover bigger classes of investors.

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As we discuss in our previous sections, it is difficult to use the SD or MV theory for the variable of profit to draw a conclusion for the variable of loss with respect to the utility function for loss. If one does not want to compare the variable of loss, we recommend converting the variable of loss to the variable of profit and apply the SD or MV theory for the variable of profit. We assume the revenue, say M, is constant, then the variable of profit, X, will be equal to $M - X^*$ as defined in (5). As the value of the constant M does not affect the results, we can set M to be any value, e.g. zero, so that $X = -X^*$ as defined in (3). However, in this example, we use the definition in (5) and conveniently set $M = 6$ so that the value of X will be from 1 to 5 while the value of X^* from 5 to 1. The profits for different locations and their ASD integrals are shown in the following table:

Place Table 5 here

In Tables 5 to 7, x represents different levels of profit with probabilities f, g and h and their corresponding ASD and DSD integrals M_n^A and M_n^D defined in (2) for $n = 1,2$ and 3 and $M = F$, G and H for different locations. Similarly, for the definition in (7), we let the ASD and DSD Integral Differentials $PQ_n^A = P_n^A - Q_n^A$ and $PQ_n^D = P_n^D - Q_n^D$ for $n = 1, 2$ and 3 and $P, Q = F$, G and H for $n = 1, 2$ and 3 and $P, Q = F$, G and H.

Place Tables 6 and 7 here

From Table 6, we find that neither F, G nor H dominates one another in the sense of FASD. However, Table 6 shows that G dominates H in the sense of SASD and, hence, G is preferred to H by the SASD risk averter. Table 6 also shows that $F \succ_3 G \succ_3 H$ and, hence, F is preferred to G which in turn is preferred to H by the TASD risk averter. From Table 7, we find that neither F, G nor H dominates one another in the sense of FDSD. However, Table 7 shows that $H \succ^2 G \succ^2 F$. Hence, H is preferred to G which in turn is preferred to F by the SDSD risk lover. Moreover, the table also shows that $H \succ^3 G \succ^3 F$ which implies that H is preferred to G which in turn is preferred to F by the TDSD risk lover.

In addition, applying the improved MV criterion, from Table 5, we have $G\ MS_A\ H$, and $H\ MS_D\ G\ MS_D\ F$. This means that G dominates H by the AMS rule while H dominates G which dominates F by the DMS rule. From Theorem 5, we conclude that if G and H belong to the same location-scale family, then G is preferred to H by the SASD risk averter. Similarly, if H, G and F belong to the same location-scale family, then H is preferred to G which in turn is preferred to F by the SDSD risk lover.

We summarize our findings from Tables 1 to 7 as follows:

A.   Using the improved Mean-Variance criterion:

MV1 $G^*\ MS_A^*\ H^*$.   That is, G^* dominates H^* by AMS rule. If both G^* and H^* belong to the same location-scale family, then G^* is preferable to H^* for the SASD risk averter.

MV1' $G\ MS_A\ H$. That is, G dominates H by AMS rule. If both G and H belong to the same location-scale family, then G is preferable to H for the SASD risk averter.

MV2 There are no preference between G^* [or H^*] and F^* in the sense of MS_A^*.

MV2' There are no preference between G [or H] and F in the sense of MS_A.

MV3 $H^*\ MS_D^*\ G^*\ MS_D^*\ F^*$. That is, H^* dominates G^* which in turn dominates F^* by the DMS rule. If H^*, G^* and F^* belong to the same location-scale family, then H^* is preferable to G^* which is preferable to F^* for the SDSD risk lover.

MV3' $H\ MS_D\ G\ MS_D\ F$. That is, H dominates G which in turn dominates F by the DMS rule. If H, G and F belong to the same location-scale family, then H is preferable to G which is preferable to F for the SDSD risk lover.

B.   Using the improved Stochastic Dominance criterion:

SD1 FSD investors have no preference among F^*, G^* and H^*;

SD1' FSD investors have no preference among *F, G* and *H*;

SD2 SASD risk averters have no preference between F^* and G^*, no preference between F^* and H^* but prefer G^* to H^*;

SD2' SASD risk averters have no preference between *F* and *G*, no preference between *F* and *H*, but prefer *G* to *H* ;

SD3 TASD risk averters prefer F^* to G^* to H^*;

SD3' TASD risk averters prefer *F* to *G* to *H;*

SD4 SDSD and TDSD risk lovers prefer H^* to G^* to F^*; and

SD4' SDSD and TDSD risk lovers prefer *H* to *G* to *F*.

From the above summary, it is easy to notice the superiority of our approaches to WKS' and DSM's as our approaches cover wider classes of investors not only including FASD and SASD risk averters as used in WKS and DSM, but also TSAD risk averters as well as FDSD, SDSD and TDSD risk lovers. Our approaches provide investors with more tools for empirical analysis, with which they can identify the first order ASD and DSD prospects and discern arbitrage opportunities that could increase his/her utility as well as wealth and set up a zero dollar portfolio to make huge profit. Our tools also could be used to identify the second and third orders ASD and DSD prospects which, in turn, enable investors to make better choices.

The above example show the superiority of using our approaches in business planning, we further illustrate the superiority of our approaches in investment by using two of the most complicated experiments in Levy and Levy (2002) - the gains one month later for an investor who invests $10,000 either in stock A or in stock B. The two experiments are shown below.

Place Experiments 1 and 2 here

We choose these two experiments from Levy and Levy because (1) these two are the most complicated examples used in their paper; and (2) the underlying SD and MV relationships in these examples are different from the commonly believed relationships between SD and MV. The examples used in most papers, for example, Weeks and Wingler (1979), WKS, Post and Diltz (1986) and DSM, include both SASD and MV dominances and match the inference drawn in Theorems 5 and 6 in our paper. However, the first experiment in Levy and Levy shows the first order SD while the second experiment shows the third order SD and its relationship with mean-variance dominance, which seems to violate the inference drawn in Theorems 5 and 6.

For both Experiments 1 and 2, let X and Y (X^* and Y^*) be the gain or profit (loss) for investing in Stocks A and B with the corresponding probability functions f and g (f^* and g^*) and the corresponding cumulative probability functions F and G (F^* and G^*) respectively. For simplicity, we only illustrate the results of variables of loss[2]. Tables 8 and 9 depict the ASD and DSD Integral Differentials for the losses of investing in Stocks A and B in Experiments 1 and 2 respectively.

Place Table 8 here

Table 8 shows that G^* dominates F^* in the sense of both FASD and FDSD, which, by applying Theorem 4, tells us that F^* is preferable to G^* for both FASD and FDSD investors. We note that hierarchy exists in SD relationships: the first order SD implies the second order SD which in turn implies the third order SD and, hence, F also dominates G in the sense of SASD, TASD, SDSD and TDSD. Thus, traditionally, we only report the lowest dominance order in practice. Table 8 also shows that the mean of F^* is smaller than that of G^* while its variance is smaller than that of G^*. This implies that F^* dominates G^* by the AMS rule which, according to Theorem 6, implies that Stock A is preferable to Stock B for the SASD risk averter.

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[2] The results of variables for profit are available on request.

For Experiment 2, Table 9 shows that F^* and G^* do not dominate each other by the FASD, SASD, TASD, FDSD or SDSD rules. But the results show that F^* dominates G^* by the TDSD and DMS rule while G^* dominates F^* by the AMS rule. All these results lead us to conclude that Stock B is preferable to Stock A by the TASD rule in terms of profit, Stock B is preferable to Stock A by the TDSD rule in terms of loss, and Stock B is preferable to Stock A by the AMS rule while Stock A is preferable to Stock B by the DMS rule.

We notice that Experiment 2 only illustrates the TASD in term of profit and TDSD in term of loss, but not TDSD in term of profit nor TASD in term of loss. In order to show examples to complete all the SD domination cases, we modify Experiment 2 by adjusting the probabilities for Stock B in Experiment 3 and obtain the results for variables of loss depicted in Tables 10[3].

Place Experiment 3, Table 10 here

For Experiment 3, Table 10 shows that F^* and G^* do not dominate each other by the FASD, SASD or FDSD rules. But the results show that F^* dominates G^* by the TASD, SDSD and TDSD rules while G^* dominates F^* by the AMS rule. All these results lead us to conclude that Stock B is preferable to Stock A by the SASD, TASD, TDSD and AMS rules in terms of profit, Stock B is preferable to Stock A by the TASD, SDSD, TDSD and AMS rules in terms of loss.

One may doubt the validity of Theorems 5 and 6 as these theorems tell us that if Stock A dominates Stock B by the AMS rule, then Stock A will dominate Stock B by the SASD rule, but Experiments 1 to 3 show that this is not true. Obtaining these apparently contradicting results is not surprising as Hanoch and Levy (1969) already state the

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[3] The results of variables for profit are available on request.

dilemma of making decisions from MV choice criteria because they contradict with that of the SD theory. Besides, Levy (1989) also shows that the MV efficient set is different from the SD efficient sets. These results cannot disprove the theorems as they require the condition of the same location-scale family. Since Stocks A and B do not belong to any location-scale family, these two theorems cannot be applied in these three experiments.

6. Conclusion

Our paper extends the work of WKS, DSM and others on Stochastic Dominance theory for both return and loss to the first three orders and link the corresponding risk-averse and risk-loving utility functions to the first three orders. Our approaches have practical value for investors as we provide investors with more tools for empirical analysis, with which they can identify the first order SD on return and loss and discern arbitrage opportunities that could increase his/her utility as well as wealth and set up a zero dollar portfolio to make huge profit. In addition, our tools also enable investors to identify the third order SD on return and loss so that they can make better choices. We also introduce the improved MV criterion to decisions in business planning for risk-averse and risk-loving investors. These new theoretically-driven methodologies enable business planners and investors to analyse many complex contemporary decision problems that could occur in various forms including inter-organizational, group-based, and technology-enabled. We illustrate the superiority of our approaches with examples from WKS, DSM and Levy and Levy (2002).

Though this paper develops methodology in SD theory for business planning and investments, our approaches could be applied not only in business planning and investment prospects, but also to many different areas in Business, Economics and Finance. For example, one could easily incorporate our approaches to explain financial theory and anomalies (see, for example, Levy and Sarnat, 1970; McNamara, 1998; Post and Levy, 2005; Kuosmanen, 2004; Fong et al., 2005). Also, one could apply our approaches to improve the models for risk management (Broll et al., 2006; Gasbarro et al, 2006) and portfolio selection problems (Soyer and Tanyeri, 2006). One could also

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incorporate our approaches to extend the theories of multiple preference comparison (see, for example, Teghem et al., 1986; Greco et al., 1999; Luciano et al., 2003).

We note that the combination of SD and MV has superiority to make better choices than by SD or by MV singly as MV criterion is easy to compute and give investors a quick review of the decision with the information of mean and variance only whereas SD could provide information for the entire distribution of each asset for comparison. Many studies have supported this viewpoint. For example, Levy (1989) and Wong and Ma (2006) have shown that the MV efficient set is different from the SD efficient sets.

In addition, we note that some authors propose to use higher order (higher than three) stochastic dominance in empirical application. For example, Vinod (2004) recommends employing the 4th order stochastic dominance to make the choice among investment prospects with illustration in his analysis of 1281 mutual funds realistic. We also note that the most commonly-used orders in stochastic dominance for empirical analyses, regardless whether they are simple or complicated, are the first three and one could easily extend the theory developed in this paper to any order. We thus stop at third order in this paper.

Finally, we summarize the decision rules obtained in this paper as follows: the first- (second-, third-) order risk-averse investors with utilities belonging to U_1^A (U_2^A, U_3^A) will choose Prospect X then Prospect Y if X dominates Y in the sense of the first- (second-, third-) order ASD and choose Loss X^* then Loss Y^* if Y^* dominates X^* in the sense of the first- (second-, third-) order ASD. The first- (second-, third-) order risk-loving investors with utilities belonging to U_1^D (U_2^D, U_3^D) will choose Prospect X then Prospect Y if X dominates Y in the sense of the first- (second-, third-) order DSD and choose Loss X^* then Loss Y^* if Y^* dominates X^* in the sense of the first- (second-, third-) order

DSD. In addition, *if X, Y, X^* and Y^** belong to the same location-scale family or the same linear combination of location-scale families, the second-order risk-averse investors with utilities belonging to U_2^A will choose Prospect X then Prospect Y if X dominates Y in the sense of the AMS rule and choose Loss X^* then Loss Y^* if X^* dominates Y^* in the sense of the AMS rule whereas the second-order risk-loving investors with utilities belonging to U_2^D will choose Prospect X then Prospect Y if X dominates Y in the sense of the DMS rule and choose Loss X^* then Loss Y^* if X^* dominates Y^* in the sense of the DMS rule.

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Table 1 : The Risk of three locations and their ASD Integrals

Costs (in million)	Probability			FASD Integrals			SASD Integrals			TASD Integrals		
x^*	f^*	g^*	h^*	F_1^{*A}	G_1^{*A}	H_1^{*A}	F_2^{*A}	G_2^{*A}	H_2^{*A}	F_3^{*A}	G_3^{*A}	H_3^{*A}
1	0	0.25	0.35	0	0.25	0.35	0	0	0	0	0	0
2	0.25	0.25	0.05	0.25	0.5	0.4	0	0.25	0.35	0	0.125	0.175
3	0.25	0	0.1	0.5	0.5	0.5	0.25	0.75	0.75	0.125	0.625	0.725
4	0.25	0	0	0.75	0.5	0.5	0.75	1.25	1.25	0.625	1.625	1.725
5	0.25	0.5	0.5	1	1	1	1.5	1.75	1.75	1.75	3.125	3.225
6	0	0	0	1	1	1	2.5	2.75	2.75	3.75	5.375	5.475
Mean	3.5	3.25	3.25									
Variance	1.25	3.18	3.39									

The ASD integral M_n^{*A} is defined in (2) for n= 1, 2 and 3; Cost x^* = 6 is included in order to measure the effect of x^* = 5 on M_2^{*A} and M_3^{*A}; $M = F, G$ and H.

Table 2 : The Risk of three locations and their DSD Integrals

Costs	FDSD Integrals			SDSD Integrals			TDSD Integrals		
x^*	F_1^{*D}	G_1^{*D}	H_1^{*D}	F_2^{*D}	G_2^{*D}	H_2^{*D}	F_3^{*D}	G_3^{*D}	H_3^{*D}
0	1	1	1	3.5	3.25	3.25	6.75	6.875	6.975
1	1	1	1	2.5	2.25	2.25	3.75	4.125	4.225
2	1	0.75	0.65	1.5	1.5	1.6	1.75	2.25	2.3
3	0.75	0.5	0.6	0.75	1	1	0.625	1	1
4	0.5	0.5	0.5	0.25	0.5	0.5	0.125	0.25	0.25
5	0.25	0.5	0.5	0	0	0	0	0	0

The DSD Integral M_n^{*D} is defined in (2) for n= 1, 2 and 3; Cost x^* = 0 is included in order to measure the effect of x^* = 1 on M_2^{*D} and M_3^{*D}; $M = F, G$ and H.

Table 3 : The ASD Integral Differentials for the Risk of three locations

x^*	HF_1^{*A}	GF_1^{*A}	HG_1^{*A}	HF_2^{*A}	GF_2^{*A}	HG_2^{*A}	HF_3^{*A}	GF_3^{*A}	HG_3^{*A}
1	0.35	0.25	0.1	0	0	0	0	0	0
2	0.15	0.25	-0.1	0.35	0.25	0.1	0.175	0.125	0.05
3	0	0	0	0.5	0.5	0	0.6	0.5	0.1
4	-0.25	-0.25	0	0.5	0.5	0	1.1	1	0.1
5	0	0	0	0.25	0.25	0	1.475	1.375	0.1
6	0	0	0	0.25	0.25	0	1.725	1.625	0.1

The ASD Integral Differential $PQ_n^{*A} = P_n^{*A} - Q_n^{*A}$ for n=1, 2 and 3; $P, Q = F, G$ and H.

Table 4 : The DSD Integral Differentials for the Risk of three locations

x^*	HF_1^{*D}	GF_1^{*D}	HG_1^{*D}	HF_2^{*D}	GF_2^{*D}	HG_2^{*D}	HF_3^{*D}	GF_3^{*D}	HG_3^{*D}
0	0	0	0	-0.25	-0.25	0	0.225	0.125	0.1
1	0	0	0	-0.25	-0.25	0	0.475	0.375	0.1
2	-0.35	-0.25	-0.1	0.1	0	0.1	0.55	0.5	0.05
3	-0.15	-0.25	0.1	0.25	0.25	0	0.375	0.375	0
4	0	0	0	0.25	0.25	0	0.125	0.125	0
5	0.25	0.25	0	0	0	0	0	0	0

The DSD Integral Differential $PQ_n^{*D} = P_n^{*D} - Q_n^{*D}$ for n=1, 2 and 3; $P, Q = F, G$ and H.

Table 5 : The Profits of three locations and their ASD Integrals

Profit (in million)	Probability			FASD Integrals			SASD Integrals			TASD Integrals		
x	f	g	h	F_1^A	G_1^A	H_1^A	F_2^A	G_2^A	H_2^A	F_3^A	G_3^A	H_3^A
1	0.25	0.5	0.5	0.25	0.5	0.5	0	0	0	0	0	0
2	0.25	0	0	0.5	0.5	0.5	0.25	0.5	0.5	0.125	0.25	0.25
3	0.25	0	0.1	0.75	0.5	0.6	0.75	1	1	0.625	1	1
4	0.25	0.25	0.05	1	0.75	0.65	1.5	1.5	1.6	1.75	2.25	2.3
5	0	0.25	0.35	1	1	1	2.5	2.25	2.25	3.75	4.125	4.225
6	0	0	0	1	1	1	3.5	3.25	3.25	6.75	6.875	6.975
Mean	2.5	2.75	2.75									
Variance	1.25	3.18	3.39									

The ASD Integral M_n^A is defined in (2) for n = 1, 2 and 3; Profit $x = 6$ is included in order to measure the effect of $x = 5$ on M_2^A and M_3^A; $M = F, G$ and H.

Table 6 : The ASD Integral Differentials for the Profits of three locations

x	HF_1^A	GF_1^A	HG_1^A	HF_2^A	GF_2^A	HG_2^A	HF_3^A	GF_3^A	HG_3^A
1	0.25	0.25	0	0	0	0	0	0	0
2	0	0	0	0.25	0.25	0	0.125	0.125	0
3	-0.15	-0.25	0.1	0.25	0.25	0	0.375	0.375	0
4	-0.35	-0.25	-0.1	0.1	0	0.1	0.55	0.5	0.05
5	0	0	0	-0.25	-0.25	0	0.475	0.375	0.1
6	0	0	0	-0.25	-0.25	0	0.225	0.125	0.1

The ASD Integral Differential $PQ_n^A = P_n^A - Q_n^A$ for $n = 1, 2$ and 3; $P, Q = F, G$ and H.

Table 7 : The DSD Integral Differentials for the Profits of three locations

x	HF_1^D	GF_1^D	HG_1^D	HF_2^D	GF_2^D	HG_2^D	HF_3^D	GF_3^D	HG_3^D
0	0	0	0	0.25	0.25	0	1.725	1.625	0.1
1	0	0	0	0.25	0.25	0	1.475	1.375	0.1
2	-0.25	-0.25	0	0.5	0.5	0	1.1	1	0.1
3	0	0	0	0.5	0.5	0	0.6	0.5	0.1
4	0.15	0.25	-0.1	0.35	0.25	0.1	0.175	0.125	0.05
5	0.35	0.25	0.1	0	0	0	0	0	0

The DSD Integral Differential $PQ_n^D = P_n^D - Q_n^D$ for $n = 1, 2$ and 3; $P, Q = F, G$ and H.

Experiment 1

Stock A		Stock B	
Gain (in thousand)	Probability	Gain (in thousand)	Probability
0.5	0.3	-0.5	0.1
2	0.3	0	0.1
5	0.4	0.5	0.1
		1	0.2
		2	0.1
		5	0.4

Experiment 2

Stock A		Stock B	
Gain (in thousand)	Probability	Gain (in thousand)	Probability
-1.6	0.25	-1	0.25
-0.2	0.25	-0.8	0.25
1.2	0.25	0.8	0.25
1.6	0.25	2	0.25

Experiment 3

Stock A		Stock B	
Gain (in thousand)	Probability	Gain (in thousand)	Probability
-1.6	0.25	-1	0.25
-0.2	0.25	-0.8	0.40
1.2	0.25	0.8	0.30
1.6	0.25	2	0.05

Table 8 : The ASD and DSD Integral Differentials for the Losses of Stocks A and B in Experiment 1

Loss(x^*)	f^*	g^*	GF_1^{*A}	GF_2^{*A}	GF_3^{*A}	GF_1^{*D}	GF_2^{*D}	GF_3^{*D}
-6	0	0	0	0	0	0	0.35	1.7875
-5	0.4	0.4	0	0	0	0	0.35	1.4375
-2	0.3	0.1	-0.2	0	0	0	0.35	0.3875
-1	0	0.2	0	-0.2	-0.1	0.2	0.15	0.1375
-0.5	0.3	0.1	-0.2	-0.2	-0.2	0	0.15	0.0625
0	0	0.1	-0.1	-0.3	-0.325	0.2	0.05	0.0125
0.5	0	0.1	0	-0.35	-0.4875	0.1	0	0
1	0	0	0	-0.35	-0.6625	0	0	0
Mean	-2.75	-2.4						
Variance	3.7125	4.89						

The ASD and DSD Integral Differentials $GF_n^{*A} = G_n^{*A} - F_n^{*A}$ and $GF_n^{*D} = G_n^{*D} - F_n^{*D}$ for n=1, 2 and 3.
Loss $x^* = 1$ is included in order to measure the effect of $x^* = 0.5$ on GF_n^{*A} while Loss $x^* = -6$ is included in order to measure the effect of $x^* = -5$ on GF_n^{*D} for n = 2 and 3.

Table 9 : The ASD and DSD Integral Differentials for the Losses of Stocks A and B in Experiment 2

$Loss(X*)$	$f*$	$g*$	GF_1^{*A}	GF_2^{*A}	GF_3^{*A}	GF_1^{*D}	GF_2^{*D}	GF_3^{*D}
-3	0	0	0	0	0	0	0	-0.04
-2	0	0.25	0.25	0	0	0	0	-0.04
-1.6	0.25	0	0	0.1	0.02	-0.25	0.1	-0.06
-1.2	0.25	0	-0.25	0.1	0.06	0	0.1	-0.1
-0.8	0	0.25	0	0	0.08	0.25	0	-0.12
0.2	0.25	0	-0.25	0	0.08	0	0	-0.12
0.8	0	0.25	0	-0.15	0.035	0.25	-0.15	-0.075
1	0	0.25	0.25	-0.15	0.005	0	-0.15	-0.045
1.6	0.25	0	0	0	-0.04	-0.25	0	0
2	0	0	0	0	-0.04	0	0	0
Mean	-0.25	-0.25						
Variance	1.5875	1.5075						

The ASD and DSD Integral Differentials $GF_n^{*A} = G_n^{*A} - F_n^{*A}$ and $GF_n^{*D} = G_n^{*D} - F_n^{*D}$ for n=1, 2 and 3.
Loss $X* = 2$ is included in order to measure the effect of x = 1.6 on GF_n^{*A} while Loss $X* = -3$ is included in order to measure the effect of $x = -2$ on GF_n^{*D} for n = 2 and 3.

Table 10 : The ASD and DSD Integral Differentials for the Losses of Stocks A and B in Experiment 3

Loss($X*$)	$f*$	$g*$	GF_1^{*A}	GF_2^{*A}	GF_3^{*A}	GF_1^{*D}	GF_2^{*D}	GF_3^{*D}
-3	0	0	0	0	0	0	-0.12	-0.436
-2	0	0.25	0.25	0	0	0	-0.12	-0.316
-1.6	0.25	0	0	0.1	0.02	-0.25	-0.02	-0.288
-1.2	0.25	0	-0.25	0.1	0.06	0	-0.02	-0.28
-0.8	0	0.3	0.05	0	0.08	0.25	-0.12	-0.252
0.2	0.25	0	-0.2	0.05	0.105	-0.05	-0.07	-0.157
0.8	0	0.4	0.2	-0.07	0.099	0.2	-0.19	-0.079
1	0	0.05	0.25	-0.03	0.089	-0.2	-0.15	-0.045
1.6	0.25	0	0	0.12	0.116	-0.25	0	0
2	0	0	0	0.12	0.164	0	0	0
Mean	-0.25	-0.37						
Variance	1.5875	1.3611						

The ASD and DSD Integral Differentials $GF_n^{*A} = G_n^{*A} - F_n^{*A}$ and $GF_n^{*D} = G_n^{*D} - F_n^{*D}$ for n=1, 2 and 3.
Loss $X* = 2$ is included in order to measure the effect of $x = 1.6$ on GF_n^{*A} while Loss $X* = -3$ is included in order to measure the effect of $x = -2$ on GF_n^{*D} for n = 2 and 3.