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Tops-Only Domains

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Abstract

In this paper we consider the standard voting model with a finite set of alternatives A and n voters and address the following question: what are the characteristics of domains $\mathcal{D}$ that induce the property that every strategy-proof social choice function $f : \mathcal{D}^n \to A$ satisfying unanimity, has the tops-only property? We first impose a minimal richness condition which ensures that for every alternative a, there exists an admissible ordering where a is maximal. We identify conditions on $\mathcal{D}$ that are sufficient for strategy-proofness and unanimity to imply tops onlyness in the general case of n voters and in the special case, $n = 2$. We provide an algorithm for constructing tops-only domains from connected graphs with elements of A as nodes. We provide several applications of our results. Finally, we relax the minimal richness assumption and partially extend our results.

1 Introduction

In the mechanism design problem, it is assumed that agents have private information but collectively wish to attain "optimal" outcomes which depend on this private information. The goal of mechanism design theory is to identify the outcomes which can be achieved (as a function of the "state of the world" or the private information held by agents) when agents are rational and fully recognize their strategic opportunities. The mapping between the profile of private information and optimal outcomes is called a *social choice function* or SCF. A natural requirement is for the SCFs to be *strategy-proof*, i.e. they should provide all agents with dominant strategy incentives to reveal their private information.

Whether or not a SCF is strategy-proof depends critically on the amount of the private information it "uses" to compute the optimal outcome. At one extreme, if an SCF is constant and does not depend at all on the information of agents, it is strategy-proof because agents

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cannot gain by misrepresenting their information. On the other hand, if the SCF depends "intricately" on private information, we imagine that it is unlikely to be strategy-proof. This is because it will afford significant opportunities for agents to benefit by lying. The models that we are concerned with (and these models are preponderant in the literature) are those where an agents' private information consists of her preferences over some fixed set of alternatives. Our goal is to investigate *domains* of preferences over which all strategy-proof SCFs use only the information of the maximal element or "top" of agent preferences in order to compute its output.

We consider the standard voting model. There are n agents called voters and a fixed and finite set of alternatives A. Each voter i has a preference P_i which is a linear ordering over the elements of A. The set of possible orderings for each agent is the set $\mathcal{D}$ which is a subset of the set of all linear orders over A. Note that we are assuming that the possible set of preferences is the same for all agents. A SCF is a mapping $f : \mathcal{D}^n \to A$. We are interested in identifying conditions on $\mathcal{D}$ with the property that *all* strategy-proof SCFs defined over this domain, satisfying the additional (weak) property of unanimity, have the tops-only property.

Tops-only SCFs are well-known in the social choice literature. The most familiar one is the complete domain or the domain of all linear orderings over A. If A has at least three elements, then the Gibbard-Satterthwaite Theorem states that every strategy-proof SCF defined over the complete domain must be *dictatorial*. A dictatorial SCF always picks the top-ranked alternative of a fixed voter called the dictator. Since the the outcome of a dictatorial SCF for every profile of preferences depends only on the top-ranked alternative of the dictator, the complete domain is a tops-only domain. The other salient tops-only domain is the domain of *single-peaked* preferences. This domain is the cornerstone of much of the modern literature on political economy. There are several variants of these domains as well as other domains which also have the tops-only property; many of them are discussed in the paper. Our objective is to identify the *common features* of these domains which precipitate the tops-only property. Our results unify a number of disparate results pertaining to specific domains and also identify some new ones. We note that if a domain can be classified as a tops-only domain, the task of characterizing strategy-proof SCFs over these domains is considerably simplified.

Most of our paper is concerned with domains which satisfy an additional property called minimal richness. Such domains have the feature that for every alternative, there is an admissible ordering where the alternative is ranked first. We prove two results for such domains. In the case of two voters, we provide an extremely simple property on domains which we call Property T that is sufficient for tops-onlyness. Unfortunately, we are unable to prove that this property is also sufficient when there are more than two voters. However we are able to provide another property called Property T^* which guarantees the tops-onlyness of domains independently of the number of voters. We show that Property T^* together with a symmetry property on domains implies Property T.

Property T is a condition formulated in terms of the notion of "connections" between alternatives originally proposed in Aswal et al. (2003). Fix a domain $\mathcal{D}$. We say that two alternatives are connected if there exists an admissible ordering where a is ranked first and b second, and another admissible ordering where b is ranked first and a second. The domain $\mathcal{D}$ satisfies Property T if every ordering $P_i \in \mathcal{D}$ has the property that every alternative that is not first-ranked in P_i is connected to an alternative that is preferred to it under P_i. Our first result states that every domain satisfying this property also satisfies the tops-only property.

Property T is easy to verify in practice. We provide an algorithm to generate domains satisfying this property from arbitrary graphs that are connected (in the conventional graph-theoretic sense). We also use this property to provide new conditions for domains to be dictatorial which are different from those in Aswal et al. (2003). In doing this, we use the well-known result that a domain that is dictatorial for $n = 2$ is also dictatorial for general n. We further use our new condition for dictatorial domains to show that such a domain must contain $2m$ orderings where $m \geq 3$ is the cardinality of $|A|$. Moreover, a domain which attains this bound is identified. This answers a question raised in Aswal et al. (2003). Recently, the same result on minimal dictatorial domains has been independently established in Sato (2008) Chapter 3.

We conjecture that Property T is sufficient for tops-onlyness generally but are unable at this time to either prove it or provide a counterexample. Our alternative Property T^* is also a fairly simple condition but does not rely on the connectivity structure of the domain. We demonstrate that salient domains such as those of single-peaked and generalized single-peaked domains and multi-dimensional domains with product structure and separable preferences (and various sub-domains of these domains) all satisfy Property T^*. Their tops-onlyness is therefore an immediate consequence of our general result. We also apply our results to models of multi-dimensional voting with constraints where preferences are additively separable. A well-known example of such a model is that of choosing committees when there are restrictions regarding their size or composition. We distinguish between two types of models here, one where the domain consists of preferences whose tops are feasible and another where no such restriction is made. We show that Property T^* is satisfied in the former case but not in the latter.

Our final section is an attempt to deal with domains which do not satisfy the minimal richness assumption. There are several well-known examples of such domains such as those where sets are ranked according to some expected utility criterion. There are serious technical difficulties in extending our results to these domains. However, we show that a suitable modification of Property T^* implies a weaker version of the tops-onlyness result which pertains only to alternatives which can be first-ranked. We apply this result to several domains. We show that in some cases, general tops-onlyness can also be obtained by exploiting specific aspects of the domains being considered.

A paper which is close to ours in spirit is Weymark (2008). It proposes a general proof technique which can be used to prove the tops-onlyness property of domains. One advantage of the approach in this paper over ours is that it also applies to models where the set of alternatives is a non-finite subset of Euclidean space. On the other hand, it does not seek to identify sufficient conditions on domains for tops-onlyness which is our primary objective. At a formal level there is no overlap between the results of the two papers. However both papers are motivated by the same issues - that of trying to understand the structure of tops-only domains and trying to unify, refine and extend a number of isolated results in this area.

The paper is organized as follows. Section 2 introduces the basic models and definitions. Section 3 presents the main results for minimally rich domains. Section 4 discusses applications of the results in Section 3 while Section 5 deals with non-minimally rich domains. Section 6 concludes.

2 Preliminaries

We let A denote a set of alternatives with $|A| = m < \infty$. The set of voters is $N = \{1, ..., n\}$. Each voter i is assumed to have a linear order P_i over the elements of the set A which we shall refer to as her *preference ordering*. For all $a, b \in A$, aP_ib will signify the statement "a is strictly preferred to b according to P_i". We let $\mathcal{P}$ denote the set of all linear orders over the elements of the set A. The set of all *admissible* ordering is a set $\mathcal{D} \subset \mathcal{P}$. A preference profile $P \equiv (P_1, ..., P_n) \in \mathcal{D}^n$ is a list of admissible preference orderings, one for each voter.

For all $k = 1, .., m$, $P_i \in \mathcal{D}$ and $a \in A$, we shall say that a is k^{th} ranked in P_i if $|\{b \in A | aP_ib\}| = m - k$. We will write $a = r_k(P_i)$ if a is k^{th} ranked in P_i. In the special case where $a = r_1(P_i)$ we will write $a = \tau(P_i)$.

The fundamental object of study in this paper is a *Social Choice Function* which we define below.

Definition **1** *A Social Choice Function (SCF) is a mapping $f : \mathcal{D}^n \to A$.*

We now introduce the important concept of a minimally rich domain. Such a domain has the property that for every a, there is a ordering P_i in the domain where a is the best alternative. Note that this implies $|\mathcal{D}| \geq m$.

Definition **2** *The domain $\mathcal{D}$ satisfies minimal richness if for all $a \in A$, there exists $P_i \in \mathcal{D}$ such that $\tau(P_i) = a$.*

We will assume in Sections 2, 3 and 4 that domains under consideration satisfy minimal richness.

We now introduce some properties of SCFs.

Definition **3** *A SCF $f : \mathcal{D}^n \to A$ satisfies unanimity if for all $P \in \mathcal{D}^n$ and $a \in A$ such that $\tau(P_i) = a$ for all $i \in N$, we have $f(P) = a$.*

If an SCF satisfies unanimity, then it always respects consensus whenever it exists, i.e. an alternative ranked first by all agents in a profile is always picked. Observe that a SCF satisfying unanimity and defined over a minimally rich domain must have a range of m.

Each voters' preference ordering is private information, i.e. known only to herself. These preferences must therefore be elicited by the mechanism designer. If a SCF is *strategy-proof*, then no voter can benefit by misrepresenting her preferences irrespective of her beliefs about the preference announcement of other voters.

DEFINITION 4 *A SCF $f : \mathcal{D}^n \to A$ is manipulable by voter i at profile $P \in \mathcal{D}^n$ via $P_i' \in \mathcal{D}$ if $f(P_i', P_{-i}) P_i f(P)$. A SCF is strategy-proof if it is not manipulable by any voter.*

It is well-known that strategy-proofness is a stringent requirement for SCFs to satisfy. In general, SCFs which "use" a lot of information are more vulnerable to manipulation because they afford voters greater opportunities for being strategic. Our goal in this paper is to investigate domains where all strategy-proof SCFs only use information regarding the most preferred alternatives of all voters.

DEFINITION 5 *The profiles $P, P' \in \mathcal{D}^n$ are tops-equivalent if $\tau(P_i) = \tau(P_i')$ for all $i \in N$. The SCF $f : \mathcal{D}^n \to A$ satisfies the tops-only property if $f(P) = f(P')$ whenever P and P' are tops equivalent.*

Several well-known SCFs satisfy the tops-only property and we will describe some of these subsequently. Our objective is to investigate domains where strategy-proofness and unanimity imply the tops only property.

DEFINITION 6 *The domain $\mathcal{D}$ satisfies the tops only property if every SCF $f : \mathcal{D}^n \to A$ which is strategy-proof and satisfies unanimity also satisfies the tops only property.*

Our objective in this paper is to provide sufficient conditions for domains to satisfy the tops-only property. In order to do so, we need to introduce some auxiliary concepts and results.

Let $\mathcal{D}$ be an arbitrary domain and let $f : \mathcal{D}^n \to A$ be an arbitrary SCF. For all $i \in N$ and $P_{-i} \in \mathcal{D}^{N-1}$, let $O_i(P_{-i}) = \{a \in A | f(P_i, P_{-i}) = a, P_i \in \mathcal{D}\}$. The set $O_i(P_{-i})$ will be referred to as the *option set* for voter i, given P_{-i} and is the restricted range of the SCF f given P_{-i}. These sets were first introduced in Barberà and Peleg (1990).

We record two elementary facts about these sets (proofs can be found in Barberà and Peleg (1990)).

- 1. For any strategy-proof SCF $f : \mathcal{D}^n \to A$, the following holds: for all $(P_i, P_{-i}) \in \mathcal{D}^n$, $f(P_i, P_{-i}) = C(P_i, O_i(P_{-i}))$.[1]

[1] For all $P_i \in \mathcal{P}$ and $B \subset A$, $C(P_i, B)$ is the maximal element in the set B according to the order P_i.

- 2. For any SCF $f : \mathcal{D}^n \rightarrow A$ satisfying unanimity, the following holds: for all $i \in N$, $P_{-i} \in \mathcal{D}^{N-1}$ and $a \in A$ such that $\tau(P_j) = a$ for all $j \neq i$, we have $a \in O_i(P_{-i})$.

A tops-only property for option sets can be defined in a natural way and shown to be equivalent to the tops-only property.

PROPOSITION **1** *The SCF f satisfies the tops-only property if and only if for all $i \in N$ and $P_{-i}, P'_{-i} \in \mathcal{D}^{n-1}$ which are tops equivalent (i.e. $P_j = P'_j$ for all $j \neq i$), we have $O_i(P_{-i}) = O_i(P'_{-i})$.*

Proof: : Suppose $O_i(P_{-i}) = O_i(P'_{-i})$ whenever P_{-i} and P'_{-i} are tops-equivalent for all $i \in N$. Let $P, P' \in \mathcal{D}^n$ be profiles which are tops-equivalent. Then, $f(P) = f(P_1, P_{-1}) = C(P_1, O_1(P_{-1})) = C(P_1, O_1((P'_{-1})) = f(P_1, P'_{-1}) = C(P'_2, O_2(P_1, P'_{-\{1,2\}})) = C(P'_2, O_2(P'_1, P'_{-\{1,2\}})) = f(P')$ which establishes that f is tops-only.

To prove the converse assume that f satisfies the tops-only property but there exists $i \in N$, $P_{-i}, P'_{-i} \in \mathcal{D}^{N-1}$ which are tops-equivalent but $O_i(P_{-i}) \neq O_i(P'_{-i})$. Assume w.l.o.g that $a \in O_i(P_{-i}) - O_j(P'_{-i})$. Since $\mathcal{D}$ is minimally rich, there exists $P_i \in \mathcal{D}$ such that $\tau(P_i) = a$. Then $f(P_i, P_{-i}) = a \neq f(P_i, P'_{-i})$ which contradicts the initial hypothesis that f is tops-only. ■

We now focus attention on the tops-only property of option sets.

3 Results

3.1 The Two-Voter Case

The central notion for our condition on admissible domains is that of *connections* which was introduced in Aswal et al. (2003).

DEFINITION **7** *Fix a domain $\mathcal{D}$. We say that alternatives $a, b \in A$ are connected if there exist $P_i, P'_i \in \mathcal{D}$ such that (i) $a = \tau(P_i)$ and $b = r_2(P_i)$ and (ii) $b = \tau(P'_i)$ and $a = r_2(P'_i)$.*

According to the definition a and b are connected if there exists an admissible ordering where a and b are ranked first and second respectively and another ordering where b and a are ranked first and second respectively. If a and b are connected, we denote it by $a \sim b$.

DEFINITION **8** *The domain $\mathcal{D}$ satisfies Property T if for all $P_i \in \mathcal{D}$ and $a \in A - \{\tau(P_i)\}$ there exists $b \in A - \{a\}$ such that (i) bP_ia and (ii) $b \sim a$.*

Property T requires the following. For any alternative in an admissible order (which is not the most preferred alternative of that order) there must exist another alternative which is better than it and to which it is connected. We will provide several instances of domains satisfying Property T in the next section.

THEOREM 1 *Assume* $n = 2$. *If a minimally rich domain satisfies Property* T, *then it satisfies the tops-only property.*

Proof: Let $\mathcal{D}$ be an arbitrary domain satisfying Property T. We will show that every SCF $f : \mathcal{D}^2 \to A$ that is strategy-proof and satisfies unanimity also satisfies the tops-only property.

We shall refer to the two voters as i and j. Using Proposition 1, it suffices to show that if $\mathcal{D}$ satisfies Property T, then $O_j(P_i) = O_j(P_i')$ whenever P_i and P_i' are tops-equivalent. Let $a \in O_j(P_i)$ where $a = r_k(P_i)$. We will show by induction on k that $a \in O_j(P_i')$.

We begin by observing that in the case where $k = 1$, $a \in O_j(P_i')$ follows because f satisfies unanimity. Now assume that $a = r_{k'}(P_i)$ and $a \in O_j(P_i)$ implies $a \in O_j(P_i')$ for all $k' < k$. We will show that $a = r_k(P_i)$ and $a \in O_j(P_i)$ implies $a \in O_j(P_i')$.

Since $\mathcal{D}$ satisfies Property T, there exists $b \in A$ such that bP_ia and $b \sim a$. We first claim that $b \in O_j(P_i)$. To see this, assume to the contrary that $b \notin O_j(P_i)$. Since $b \sim a$, there exists $P_j \in \mathcal{D}$ such that $b = \tau(P_j)$ and $a = r_2(P_j)$. Using fact 1 and the assumptions that $b \notin O_j(P_i)$ and $a \in O_j(P_i)$, it follows that $f(P_i, P_j) = a$. Let $P_i' \in \mathcal{D}$ be such that $\tau(P_i') = b$ (again feasible because of the minimal richness assumption on $\mathcal{D}$). By unanimity, $f(P_i', P_j) = b$. But then since bP_ia, i manipulates at (P_i, P_j) via P_i', contradicting the strategy-proofness of f. Hence $b \in O_j(P_i)$.

Since bP_ia, it must be the case that $b = r_{k'}(P_i)$ for some $k' < k$. Therefore the induction hypothesis implies that $b \in O_j(P_i')$.

We now claim that $a \in O_j(P_i')$. Suppose that this is false. Since $b \sim a$, there exists $P_j' \in \mathcal{D}$ such that and $a = \tau(P_j')$ and $b = r_2(P_j')$. Since $a \in O_j(P_i) - O_j(P_i')$ and $b \in O_j(P_i')$, it follows that $f(P_i, P_j') = a$ and $f(P_i', P_j') = b$. Since bP_ia, i will manipulate f at (P_i, P_j') via P_i' contradicting the strategy-proofness of f. Hence $a \in O_i(P_i')$. An identical argument with i and j interchanged establishes that $O_i(P_j) = O_i(P_j')$ whenever P_j and P_j' are tops-equivalent. Hence f satisfies the tops-only property. ■

OBSERVATION 1 The proof of Theorem 1 also reveals the structure of option sets of strategy-proof SCFs defined over domains satisfying Property T. For any $P_i \in \mathcal{D}$, $O_j(P_i)$ consists of a *collection of chains*. A chain is a sequence $\{a_1, a_2, ..., a_K\}$ where $a_1 = \tau(P_i)$, $a_kP_ia_{k+1}$ and $a_k \sim a_{k+1}$ for $k = 1, ..., K-1$. A critical property of the two-voter setting is that the peak is always a member of the option set (guaranteed by unanimity) and is the first element of the chain. The complications of the more than two voter case arise precisely because this property holds only when there are two voters.

3.2 The General Case

We are unable to prove that Property T is sufficient for $\mathcal{D}$ to satisfy the tops-only property when there are three or more voters. However, we are able to identify a stronger condition, which we call Property T^*, that is sufficient independently of the number of voters.

For all $P_i \in \mathcal{D}$ and $a \in A - \tau(P_i)$, let $B(P_i, a) = \{x \in A | xP_ia\}$ and $W(P_i, a) = \{x \in A | aP_ix\}$. Thus, $B(P_i, a)$ and $W(P_i, a)$ are the sets of elements which are respectively better than and worse than a according to P_i.

Let $\bar{B}(P_i, a) = \{x \in A | x \in B(P_i', a)$ for all P_i' such that $\tau(P_i) = \tau(P_i')\}$. This set consists of alternatives that are better than a in all orderings P_i' that are tops equivalent to P_i. Note that $\bar{B}(P_i, a) \neq \emptyset$, since $\tau(P_i) \in \bar{B}(P_i, a)$.

Definition **9** *Fix a domain $\mathcal{D}$. Let $a \in A$ and $P_i \in \mathcal{D}$ be such that $a \neq \tau(P_i)$. Then a is satisfactory for P_i if for all $x \in \bar{B}(P_i, a)$ there exists $\bar{P}_i$ such that $\tau(\bar{P}_i) = a$ and $x\bar{P}_iW(P_i, a)$. Moreover $\mathcal{D}$ satisfies Property T^* if for all $P_i \in \mathcal{D}$ and $a \in A - \tau(P_i)$, a is satisfactory for P_i.*

Property T^* can be thought of as a sort of *reversality* property. Let P_i be an admissible ordering and let a be an alternative distinct from $y = \tau(P_i)$. Let $\bar{B}(P_i, a)$ denote the alternatives which are always better than a in *all* orderings where y is maximal. Pick an arbitrary alternative $x \in \bar{B}(P_i, a)$. Property T^* postulates the existence of an ordering where a is maximal and x is preferred to all alternatives that a was better than under P_i.

In the next section we will provide several examples of domains that satisfy Property T^*. Here we comment on the relationship between Properties T and T^*. We first show that Property T^* together with a symmetry property implies Property T.

Definition **10** *The domain $\mathcal{D}$ satisfies link symmetry if, for all $a, b \in A$, $a \sim b$ whenever there exists $P_i \in \mathcal{D}$ such that $\tau(P_i) = a$ and $r_2(P_i) = b$.*

This symmetry condition implies that whenever there exists an admissible ordering where a and b are first and second ranked respectively, there exists another ordering where b and a are first and second ranked respectively.

Proposition **2** *Let $\mathcal{D}$ be a domain satisfying link symmetry and Property T^*. Then it satisfies Property T.*

Proof: : Let $\mathcal{D}$ be a domain satisfying link symmetry and Property T^*. Let $P_i \in \mathcal{D}$ and $a \in A - \tau(P_i)$. Pick an arbitrary $x \in \bar{B}(P_i, a)$. From Property T^*, it follows that there exists $\bar{P}_i \in \mathcal{D}$ such that $\tau(\bar{P}_i) = a$ and $x\bar{P}_i W(P_i, a)$. Let $y = r_2(\bar{P}_i)$. By link symmetry y is connected to a. Note that either $y = x$ or $y\bar{P}_i x$ since $y \in W(P_i, a)$ implies that $x\bar{P}_i W(P_i, a)$ is violated. Therefore $y \in B(P_i, a)$. Hence Property T is satisfied. ■

Thus Property T^* together with link symmetry implies Property T. Property T does not, of course, imply Property T^*. In the section on constrained voting, we will provide an example of a well-behaved domain which satisfies Property T but violates Property T^*. Finally, we identify a property expressed in terms of linkages which implies both Properties T and T^*. Consider the following property: for all $P_i \in \mathcal{D}$, for all $a \in A - \tau(P_i)$ and for all $x \in \bar{B}(P_i, a)$, we have $x \sim a$. This property clearly implies both Properties T^* and T. However, we do not work with this property because it is rather strong and several domains which satisfy Properties T and T^* do not satisfy it.

Our main result in this section is the following.

Theorem 2 *If a minimally rich domain satisfies property T^*, then it satisfies the tops-only property.*

Proof: : We will prove the result by induction on the number of voters n. Let $\mathcal{D}$ be a domain satisfying Property T^*. Since SCFs under consideration satisfy unanimity, the result is trivially true for the case $n = 1$. Assume now that for some arbitrary integer n, every SCF $f : \mathcal{D}^n \to A$ which is strategy-proof and satisfies unanimity, also satisfies the tops-only property. We will show that an arbitrary SCF $f : \mathcal{D}^{n+1} \to A$ which is strategy-proof and satisfies unanimity, also satisfies the tops-only property. In view of Proposition 1, it will be sufficient to show the following. Let i be an arbitrary voter and let $P_{-i}, P'_{-i} \in \mathcal{D}^n$ be two tops equivalent profiles; then $O_i(P_{-i}) = O_i(P'_{-i})$. We proceed in four steps.

Step 1: Pick $j \neq i$ and let $P_i, P'_i \in \mathcal{D}$ be tops equivalent. Then $C(P_i, O_i(P_i, P_{-\{i,j\}})) = C(P'_i, O_i(P'_i, P_{-\{i,j\}}))$.

Define the n-voter SCF $g : \mathcal{D}^n \to A$ as follows: for all $P_i \in \mathcal{D}$ and $P_{-\{i,j\}} \in \mathcal{D}^{n-2}$, $f(P_i, P_i, P_{-\{i,j\}}) = g(P_i, P_{-\{i,j\}})$. In other words, g is obtained by "coalescing" voters i and j. It is easy to verify that g is strategy-proof and satisfies unanimity (details may be found in Sen (2001). The induction hypothesis therefore implies that g satisfies the tops-only property. Since P_i and P'_i are tops equivalent, it follows that $C(P_i, O_i(P_i, P_{-\{i,j\}})) = f(P_i, P_i, P_{-\{i,j\}}) = g(P_i, P_{-\{i,j\}}) = g(P'_i, P_{-\{i,j\}}) = f(P'_i, P'_i, P_{-\{i,j\}}) = C(P'_i, O_i(P'_i, P_{-\{i,j\}})$. This completes Step 1.

Step 2: Let $x^* = f(P_i, P_i, P_{-\{i,j\}})$ and let $a \in O_i((P_i, P_{-\{i,j\}}))$ be such that $a \neq x^*$. Then $x^* \in \bar{B}(P_i, a)$.

Suppose that the claim in Step 2 is false. Then there exists $P'_i \in \mathcal{D}$ such that P_i and P'_i are tops equivalent and $aP'_i x^*$. By Step 1, $f(P'_i, P'_i, P_{-\{i,j\}}) = x^*$. Let $y = f(P'_i, P_i, P_{-\{i,j\}}) =$

$C(P'_i, O_i(P_i, P_{-\{i,j\}}))$. Since $a \in O_i((P_i, P_{-\{i,j\}}))$, it must be the case that either $y = a$ or $yP'_i a$. Clearly voter j will manipulate in profile $(P'_i, P'_i, P_{-\{i,j\}})$ via P_i. This completes Step 2.

Step 3: $O_i(P_j, P_{-\{i,j\}}) = O_i(P'_j, P_{-\{i,j\}})$ whenever P_j and P'_j are tops equivalent.

Suppose that the claim in Step 3 is false. We can assume without loss of generality that there exists $a \in O_i(P_j, P_{-\{i,j\}}) - O_i(P'_j, P_{-\{i,j\}})$. Suppose $a = \tau(P_j) = \tau(P'_j)$. Pick $P_i \in \mathcal{D}$ such that $\tau(P_i) = a$. Then $f(P_i, P_j, P_{-\{i,j\}}) = a$ but $f(P_i, P'_j, P_{-\{i,j\}}) = b \neq a$. Since $\tau(P'_j) = a$, we have $aP'_j b$ and voter j will manipulate at $(P_i, P'_j, P_{-\{i,j\}})$ via P_i.

Assume therefore that $a \neq \tau(P_j)$. Let $x^* = f(P_j, P_j, P_{-\{i,j\}})$. It follows from Step 2 that either $a = x^*$ or $x^* \in \bar{B}(P_j, a)$. If $a = x^*$, then $a \in O_i(P'_j, P_{-\{i,j\}})$ since $f(P_j, P_j, P_{-\{i,j\}}) = f(P'_j, P'_j, P_{-\{i,j\}})$. Suppose therefore that $a \neq x^*$. Since $\mathcal{D}$ satisfies Property T^*, there exists $\bar{P}_j \in \mathcal{D}$ such that $\tau(\bar{P}_i) = a$ and $x^* \bar{P}_j W(P_j, a)$. Then $f(\bar{P}_j, P_j, P_{-\{i,j\}}) = C(\bar{P}_j, O_i(P_j, P_{-\{i,j\}})) = a$. Let $f(\bar{P}_j, P'_j, P_{-\{i,j\}}) = C(\bar{P}_j, O_i(P'_j, P_{-\{i,j\}})) = y$. Since $x^* \in O_i(P'_j, P_{-\{i,j\}}))$ (this follows easily from Step 1) and $x^* W(P_j, a)$, it follows that $yP_j a$. Hence voter j manipulates at $(\bar{P}_j, P_j, P_{-\{i,j\}})$ via P'_j. This completes Step 3.

Step 4: $O_i(P_{-i}) = O_i(P'_{-i})$ whenever $P_{-i} = P'_{-i}$ are tops equivalent.

This follows from repeated application of Step 3. ■

We now discuss issues pertaining to the necessity of Properties T and T^*.

3.3 Necessity

Property T^* is not necessary for the tops-only property to hold. We provide an example showing this in Section 4.6.2 (Example 5). However we show below that if $m \geq 3$ it is possible to construct non tops-only domains which violate both Property T and T^*.

Example **1** Let $A = B \cup C$, $B \cap C = \emptyset$. Let $B = \{b_1, b_2, ..., b_{m_1}\}$ and $C = \{c_1, c_2, ..., c_{m_2}\}$ and assume $m_1 \geq 2$. Let $\hat{\mathcal{D}}$ be the largest domain of orderings P_i which satisfies the restriction that either $b_j P_i c_j$ for all $b_j \in B$ and $c_j \in C$ or $c_j P_i b_j$ for all $b_j \in B$ and $c_j \in C$. In other words, either all alternatives in B are above all alternatives in C or vice-versa. Observe that $|\hat{\mathcal{D}}| = 2m_1!m_2!$. Observe also that all alternatives in B are mutually connected; the same is true for all alternatives in C. However alternatives in B are not connected to alternatives in C. This immediately implies that Property T is violated. Since link symmetry is satisfied by $\hat{\mathcal{D}}$, Property T^* is also violated. We claim that $\hat{\mathcal{D}}$ is not a tops-only domain. To see this consider the case of $n = 2$ and the following SCF: voter 2 chooses the maximal alternative from either B or C depending on which one of these sets is on "top" of voter 1's preference ordering. It is straightforward to verify that the SCF is strategy-proof. Let P_1 be an ordering whose top elements are from B. Let P_2 be the ordering $c_1 P_2 ... P_2 c_{m_2} P_2 b_1 P_2 .... P_2 b_{m_1}$ and let

P'_2 be the ordering $c_1P'_2...P'_2c_{m_2}P'_2b_2, P'_2....P'_2b_{m_1}$. Clearly the profiles (P_1, P_2) and (P_1, P'_2) are tops-equivalent. However $f(P_1, P_2) = b_1$ and $f(P_1, P'_2) = b_2$ violating tops-onlyness.

We now turn to applications of our results.

4 Applications

4.1 An Algorithm for Generating Tops-Only Domains for $n = 2$ from Graphs

We first show how Property T in conjunction with an arbitrary connected graph with m nodes can be used to generate domains satisfying the tops-only property in the case of two voters.

Let G be a graph with m nodes labeled $\{a_1, .., a_m\}$. We restrict attention to *connected graphs*, i.e. to graphs which have the property that every pair of nodes in G, is connected by a sequence of edges in G. Domains $\mathcal{D}$ satisfying Property T can be constructed from G by using G in the following manner: $a_j \sim a_k$ iff a_j and a_k are connected by an edge in G.

The following algorithm can be used. Pick an arbitrary a_j. For every a_k such that a_j and a_k are connected by an edge, create an ordering where a_j is first and a_k is second. The third ranked alternative in the ordering is an alternative which is connected by an edge to either a_j or a_k. The ordering is completed by ensuring that if a_r is ranked k, there is an alternative a_s which is ranked k' with $k' < k$ and a_r and a_s are connected by an edge in G. It is easy to verify that such an ordering can always be constructed if G is a connected graph.

We clarify the algorithm with the help of the following example.

Example **2** Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and let G be the graph in Figure 1. Let P_1 be an ordering consistent with G whose first-ranked alternative is a_1. Since the only edge in G connecting a_1 is one which connects it to a_2, $r_2(P_1) = a_2$. By a similar argument, $r_3(P_1) = a_3$. The fourth ranked alternative can either be a_6 or a_4 and so on. Adding orderings with each alternative as first ranked and proceeding as above, we can ensure that Property T is satisfied.

Let $\mathcal{D}$ comprise the orderings $\{P_1, P_2, ..., P_{12}\}$ which have been constructed using the algorithm.

It is easily verified that $a_j \sim a_k$ for all $a_j, a_k \in A$ if and only if a_j and a_k are connected by an edge in G. Clearly $\mathcal{D}$ satisfies Property T. Therefore Theorem 1 implies that all two-person SCFs defined over $\mathcal{D}$ satisfies the tops-only property.

Is $\mathcal{D}$ dictatorial? [2] Interestingly, the answer is no. Consider the following two-person SCF defined over this domain. Voter 1 gets his first-ranked alternative in all profiles except

[2]A formal definition is provided in the next subsection, Definition 12.

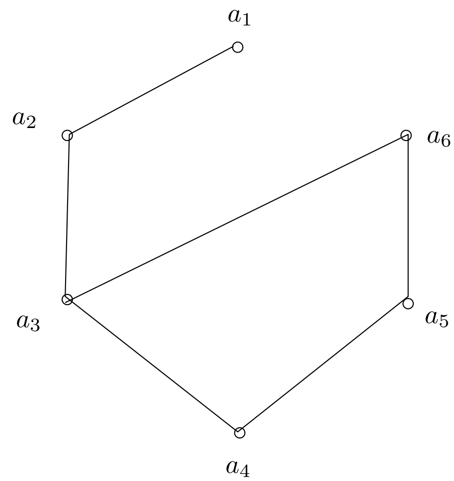

Figure 1: A Connectivity Graph

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
a_1	a_2	a_2	a_3	a_3	a_3	a_4	a_4	a_5	a_5	a_6	a_6
a_2	a_1	a_3	a_2	a_6	a_4	a_5	a_5	a_6	a_4	a_3	a_5
a_3	a_3	a_4	a_6	a_5	a_2	a_3	a_3	a_3	a_3	a_2	a_3
a_6	a_4	a_6	a_4	a_4	a_1	a_2	a_2	a_4	a_2	a_1	a_2
a_5	a_5	a_5	a_5	a_2	a_5	a_6	a_6	a_2	a_1	a_5	a_3
a_4	a_6	a_1	a_1	a_1	a_6	a_1	a_1	a_1	a_6	a_4	a_1

Table 1: The domain $\mathcal{D}$

those in which she ranks a_1 first. In this case the SCF selects voter 2's best alternative from the set $\{a_1, a_2\}$. Voter 2 clearly cannot manipulate. Voter 1 gets her second best alternative when her ordering is P_1; however there is no way for her to get her best alternative a_1 since a_2 beats a_1 for all admissible orderings. More subtly, this SCF satisfies the tops-only property.

Finally note that there are other domains which can be constructed from G by the algorithm. For instance, let P_{13} be the ordering: $a_6 P_{13} a_3 P_{13} a_5 P_{13} a_2 P_{13} a_1 P_{13} a_4$ and let $\mathcal{D}' = \mathcal{D} \cup \{P_{13}\}$. Then the connectivity structure of alternatives in $\mathcal{D}'$ is consistent with that of G and it also satisfies Property T. Hence $\mathcal{D}'$ is also a tops-only domain.

4.2 Dictatorial Domains

We now consider applications pertaining to dictatorial domains.

Definition **11** *A SCF $f : \mathcal{D}^n \to A$ is dictatorial if there exists a voter i such that for all $P \in \mathcal{D}^n$, $f(P) = \tau(P_i)$.*

Definition **12** *The domain $\mathcal{D}$ is dictatorial if every strategy-proof SCF $f : \mathcal{D}^N \rightarrow A$ satisfying unanimity is dictatorial.*

Definition **13** *The domain $\mathcal{D}$ is a minimal dictatorial domain if it is dictatorial and $|\mathcal{D}| \leq |\mathcal{D}'|$ for every dictatorial domain $\mathcal{D}'$.*

A dictatorial SCF obviously satisfies the tops-only property and a dictatorial domain is therefore a tops-only domain. We consider two variants of Property T and examine dictatorial domains using them. An interesting feature of these conditions is that these are respectively a weakening and a strengthening of the two-voter condition for tops-onliness rather than the three or more condition (i.e. Property T rather than Property T^*). The reason for this is that the extension from two to multi-voter settings for dictatorial domains (satisfying minimal richness) comes for free.

We now turn to a strengthening of Property T that yields a sufficient condition for a domain to be dictatorial. This new condition is independent of the condition of linked domains and yields new results on dictatorial domains. Indeed we are able to solve the question of identifying a minimal dictatorial domain raised in Aswal et al. (2003), with our condition.

Definition **14** *Let $B \subset A$ such that $m > |B| > 1$ and let $a \in B$. The domain $\mathcal{D}$ satisfies Property T' if there exists $\bar{P}_i, P_i' \in \mathcal{D}$ and $c, d \in A$ such that (i) $c \in B$, $d \in A - B$ and $c \sim d$ and (ii) $\tau(\bar{P}_i) = \tau(P_i') = a$, $c\bar{P}_i d$ and $dP_i' c$.*

Property T' expresses another reversality property. Pick a partition $(B, A - B)$ of the set A such that B has at least two elements and $A - B$ at least one and let $a \in B$. Then, there exists $c \in B$ and $d \in A - B$ which are connected and for which an appropriate reversal exists; in particular, there exists admissible orderings which have a as the peak and for which the preferences for c and d are reversed. Informally, every non-trivial partition of A must have a reversal. Our main result of this subsection is that Property T' in conjunction with Property T precipitates dictatorship provided $m \geq 3$. Note that Property T' is trivially satisfied if $m < 3$.

Definition **15** *The domain $\mathcal{D}$ satisfies Property T^d if it satisfies both Properties T and T'.*

We now state the main result of this subsection.

Theorem **3** *Assume $m \geq 3$. If a domain satisfies T^d, then it is dictatorial.*

Proof: : In view of the results in Aswal et al. (2003),Kim and Roush (1989), it suffices to prove the result in the case of two voters, i.e. to prove that if $\mathcal{D}$ satisfies Property T^d, then $f : \mathcal{D}^2 \rightarrow A$ is strategy-proof and satisfies unanimity implies that f is dictatorial. We assume therefore that $n = 2$ (the set of voters will be denoted by $N = \{i, j\}$) and that $\mathcal{D}$ satisfies Property T^d. Let $f : \mathcal{D}^2 \to A$ be a strategy-proof SCF satisfying unanimity.

Suppose that the Theorem is false. Since $m \geq 3$, we can assume w.l.o.g that there exists $P_i \in \mathcal{D}$ and $m > O_j(P_i) > 1$. Let $B = O_j(P_i)$ and $a = \tau(P_i)$. Note that $a \in O_j(P_i)$ since f satisfies unanimity. Let $c, d \in A$ and $\bar{P}_i, P_i' \in \mathcal{D}$ be as specified in the Definition 13. Since $\mathcal{D}$ satisfies Property T, Theorem 1 implies that $O_j(P_i) = O_j(\bar{P}_i) = O_j(P_i')$. Now pick $P_j \in \mathcal{D}$ such that $\tau(P_j) = d$ and $r_2(P_j) = c$ (this is feasible because $c \sim d$ by assumption). Also since $d \notin O_j(P_i)$ and $c \in O_j(P_i)$, we have $f(\bar{P}_i, P_j) = f(P_i', P_j) = c$. Let $P_i^* \in \mathcal{D}$ be such that $\tau(P_i^*) = d$. Then unanimity implies $f(P_i^*, P_j) = d$. Since $dP_i'c$ by assumption, voter i manipulates f at (P_i', P_j) via P_i^* which contradicts the assumption that f is strategy-proof. ■

Observation **2** Property T^d is distinct from the *linked domain* property in Aswal et al. (2003). In order to see this it is sufficient to note that the definition of a linked domain depends entirely on the connectivity relation unlike T^d while Property T^d requires restrictions other than connections. It is possible to provide a weaker (but less transparent) version of Property T and Property T^d which generalizes the Aswal et al. (2003) result. We do not pursue this line of enquiry further in this paper.

We now provide several applications of Theorem 3.

4.2.1 Free Pair at the Top and Complete Domains

A free pair at the top or FPT domain is a domain where every pair of alternatives are connected. Note that such domains can be much smaller than the universal domain $\mathcal{P}$.

Definition **16** *The domain $\mathcal{D}$ satisfies the free pair at the top domain (FPT) if $a \sim b$ for all $a, b \in A$.*

It is easy to verify that the FPT domain satisfies Property T^d. We only show that this domain satisfies Property T'. Let $B \subset A$ with $m > |B| > 1$ and let $a \in B$. Now pick any $c \in B - \{a\}$ and $d \in A - B$. Let $\bar{P}_i$ and P_i' be such that $\tau(\bar{P}_i) = \tau(P_i') = a$, $r_2(\bar{P}_i) = c$ and $r_2(P_i') = d$. Clearly c, d and $\bar{P}_i, P_i'$ satisfy the requirements of Property T'. Since $\bar{P}_i$ and P_i' belong to the FPT domain, this domain satisfies Property T'. Hence we have

Corollary **1** *Aswal et al. (2003), Gibbard (1973), Satterthwaite (1975) . Let $m \geq 3$. The FPT and complete domains are dictatorial.*

4.2.2 Circular Domains

It will be convenient to write $A = \{a_1, a_2, .., a_m\}$. The idea is to write the alternatives $a_1, a_2, .., a_m$ on the circumference of a circle. There are two orders with a_j as the first-ranked alternative. Either move clockwise from a_j so that a_{j+1} is the second-ranked alternative and a_{j-1} is the worst alternative or move counter-clockwise from a_j so that a_{j-1} is the second-ranked alternative and a_{j+1} is the worst alternative. The construction is very similar to single-peaked preferences with one critical difference: starting from any alternative, by moving either "left" or "right", one reaches a "neighbour" of the alternative again. Formally, we have the following.

Definition **17** *The Circular Domain $\mathcal{D}^C$ consists of the following $2m$ orders:* $\{a_j a_{j+1}...a_m a_1..a_{j-1}, a_j a_{j-1}...a_1 a_m...a_{j+1}, j = 1, ..., m\}$. *Here a_{m+1} is the alternative a_1.*

We illustrate this definition with an example.

Example **3** Let $A = \{a_1, a_2, a_3, a_4\}$. Then $\mathcal{D}^C$ consists of the following eight orderings $(P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8)$:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_4	a_3	a_1	a_4	a_2	a_1	a_3
a_3	a_3	a_4	a_4	a_1	a_1	a_2	a_2
a_4	a_2	a_1	a_3	a_2	a_4	a_3	a_1

Table 2: A Circular Domain

Corollary **2** *Let $m \geq 3$. The domain $\mathcal{D}^C$ is dictatorial.*

Proof: : Consider the orders $P_i = a_j a_{j+1}...a_m a_1..a_{j-1}$ and $P_i' = a_j a_{j-1}\ ...a_1 a_m...a_{j+1}$ for some $j = 1, .., m$ and a_{m+1} understood to being the alternative a_1. Observe that $r_k(P_i) \sim r_{k-1}(P_i)$ and $r_k(P_i') \sim r_{k-1}(P_i')$ for all $k = 2, .., m$. Hence $\mathcal{D}^C$ satisfies Property T.

We now show that it satisfies Property T'. Pick an arbitrary $B \subset A$ with $m > |B| > 1$ and let $a_j \in B$. Since $\{B, A - B\}$ is a partition of points that lie on the circumference of a circle, there must a pair of adjacent points which lie in different elements of the partition, i.e. there exists $k \neq j$ such that either $a_k \in B$ and $a_{k+1} \in A - B$ or $a_k \in A - B$ and $a_{k+1} \in A$. We have already seen that $a_k \sim a_{k+1}$. Observe that in P_i, we have $a_{k+1} P_i a_k$ while in P_i', we have $a_k P_i' a_{k+1}$. Hence Property T' is also satisfied. ■

Our next result shows that the circular domain is a salient dictatorial domain. [3]

[3] This result has been independently obtained by Shin Sato (Sato (2008)).

THEOREM 4 *The circular domain is a minimal dictatorial domain. If $\mathcal{D}$ is a minimal dictatorial domain, then $|\mathcal{D}| = 2m$.*

Proof: : Since $\mathcal{D}^C$ is dictatorial and has a cardinality of $2m$, we know that the cardinality of a minimal dictatorial domain is no greater than $2m$. We now show that there does not exist a minimally rich dictatorial domain whose cardinality is strictly less than $2m$. Suppose such a domain $\mathcal{D}$ existed. Since $\mathcal{D}$ satisfies minimal richness, there must exist an alternative $a \in A$ and an order $P_i \in \mathcal{D}$ such that $\tau(P_i) = a$ and $\tau(P_i') \neq a$ for all $P_i' \in \mathcal{D} - \{P_i\}$. Let $b = r_2(P_i)$. Define the SCF $f : \mathcal{D}^N \to A$ as follows. For all $P' \in \mathcal{D}^N$, such that $P_1' \neq P_i$, $f(P') = \tau(P_1')$ and for all other profiles $\hat{P}$, $f(\hat{P}) = C(\{a, b\}, \hat{P}_2)$. It is easy to verify that f is strategy-proof, satisfies unanimity and is not dictatorial. Hence $\mathcal{D}$ is not a dictatorial domain. ■

OBSERVATION 3 Aswal et al. (2003) proved that if $\mathcal{D}$ was a minimal dictatorial domain, then $|\mathcal{D}| \leq 4m - 6$. In this paper we prove a tight bound $2m$ which is, of course, strictly less than $4m - 6$ for $m > 3$. Note that for $m = 3$, the FPT, circular and complete domains all coincide.

4.3 Single-Peaked Domains

These domains have been extensively studied (see for example Moulin (1980)). We are able to apply our results to these domains.

DEFINITION 18 *Let $>$ be a linear ordering over A. A preference ordering P_i is Single-Peaked (with respect to $>$) if for all $a, b \in A$, $[\tau(P_i) > a > b$ or $b > a > \tau(P_i)] \Rightarrow aP_ib$.*

Alternatives are ordered, say on the real line. An ordering is single-peaked if alternative a which lies "between" the peak of the ordering and another alternative b, is strictly preferred to b. We will let $\mathcal{D}^{SP}$ denote the set of all single-peaked preferences with respect to some fixed order $>$.

Let $a, b \in A$ and assume that $a > b$. We say that a and b are *adjacent* if there does not exist c distinct from a and b such that $a > c > b$. Also let $[a, b]$ denote the alternatives which lie "between" a and b, i.e $\{c | a > c > b\}$ if $a > b$ and $\{c | b > c > a\}$ if $b > a$. It is easy to verify the following fact which we record as an Observation.

OBSERVATION 4 Let $\mathcal{D} \subset \mathcal{D}^{SP}$. Then, for all $a, b \in A$, $[a \sim b] \Rightarrow [a$ and b are adjacent$]$.

We will say that a domain is *adjacency rich* if the reverse implication also holds.

DEFINITION 19 *The domain $\mathcal{D} \subset \mathcal{D}^{SP}$ is adjacency rich if for all $a, b \in A$, $[a$ and b are adjacent $] \Rightarrow [a \sim b]$.*

We illustrate these concepts with an example.

EXAMPLE **4** Let $A = \{a, b, c, d\}$ and let $a > b > c > d$. Note that $\{a, b\}$, $\{b, c\}$ and $\{c, d\}$ are adjacent. Let $\mathcal{D} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ where

P_1	P_2	P_3	P_4	P_5	P_6
a	b	b	c	c	d
b	a	c	d	b	c
c	c	a	b	a	b
d	d	d	a	d	a

Then $\mathcal{D}$ is an adjacency rich single-peaked domain.

Note that adjacency rich domains can be strict subsets of $\mathcal{D}^{SP}$. For instance in Example 4 above, the single-peaked orderings *bcda* and *cbda* are absent. In general $|\mathcal{D}| = 2^{m-1}$ while the minimal adjacency rich domain has size $2^{m-2} + 2$. Note also that the superset of an adjacency rich domain is also adjacency rich. Hence $\mathcal{D}^{SP}$ is adjacency rich.

PROPOSITION **3** *Assume $n = 2$. Then every adjacency rich domain $\mathcal{D} \subset \mathcal{D}^{SP}$ satisfies the tops-only property.*

Proof: . Suppose $\mathcal{D}$ be adjacency rich. Pick $P_i \in \mathcal{D}$ and $a \in A - \tau(P_i)$. Since P_i is single-peaked, there exists $b \in A$ such that a and b are adjacent and bP_ia. By adjacency richness, $a \sim b$ so that $\mathcal{D}$ satisfies Property T. The result now follows by applying Theorem 1. ■

We prove a weaker result in the case where there are at least three voters.

PROPOSITION **4** *Assume $n \geq 3$. The domain $\mathcal{D}^{SP}$ satisfies the tops-only property.*

Proof: : Assume without loss of generality that $A = \{a_1, a_2, ..., a_M\}$ and that $a_1 > a_2 > ... > a_M$. We have seen that $a_j \sim a_k$ only if j and k are consecutive integers. Let $P_i \in \mathcal{D}^{SP}$ and let $\tau(P_i) = a_j$. Pick $a_k \neq a_j$. Let $r \in [j, k]$. Observe that for all single-peaked P'_i for which $\tau(P'_i) = a_j$, we must have $a_r P'_i a_k$. Moreover, for any integer $s \notin [j, k]$, there exists a single-peaked ordering $\bar{P}_i$ such that $\tau(\bar{P}_i) = a_j$ and $a_k \bar{P}_i a_s$. Hence $\bar{B}(P_i, a_k) = \{a_j, a_{j+1}, ..., a_{k-1}, a_k\}$ if $k > j$ and $\bar{B}(P_i, a_k) = \{a_j, a_{j-1}, .., a_{k+1}, a_k\}$ if $j > k$.

Suppose $k > j$. Pick $a_s \in \bar{B}(P_i, a_k)$. The ordering P'_i where $a_k P'_i a_{k-1} ... P'_i a_s, .., P'_i a_1 P'_i a_{k+1}, ..., P'_i a_M$ is single-peaked. If $j > k$ and $a_s \in \bar{B}(P_i, a_k)$, we can choose a single-peaked P'_i such that $a_k P'_i a_{k+1} ... P'_i a_s, .., P'_i a_M P'_i a_{k-1}, ..., P'_i a_1$. These observations establish that $\mathcal{D}^{SP}$ satisfies Property T^*. We now apply Theorem 2. ■

OBSERVATION **5** There are single-peaked domains which are strictly contained in $\mathcal{D}^{SP}$ which also satisfy Property T^*. An instance of this is the domain in Example 4.

4.4 Generalized Single-Peaked Domains

The notion of single-peakedness has been extended by Nehring and Puppe (2007b), Nehring and Puppe (2007a) to apply to sets of alternatives with a more general structure than that of the real line. We follow their formulation here.

Definition **20** *A Property Space is a pair $(A, \mathcal{H})$ where $\mathcal{H}$ is a collection of subsets of A satisfying*

1. $\emptyset \notin \mathcal{H}$
2. $H \in \mathcal{H} \Rightarrow (A - H) \in \mathcal{H}$
3. *For all $x \neq y$, there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$.*

A Property Space $(A, \mathcal{H})$ induces a ternary *betweenness relation* $B \subset A^3$ according to

$$(a, b, c) \in B \;\Leftrightarrow [\text{for all } H \in \mathcal{H} : \{a, c\} \subseteq H \Rightarrow b \in H].$$

Definition **21** *Let $(A, \mathcal{H})$ be a Property Space inducing the betweenness relation B. The ordering P_i is Generalized Single-Peaked (GSP) if for all $a, b \in A$, $(\tau(P_i), a, b) \in B \Rightarrow aP_ib$.*

We can easily generalize the notion of adjacency richness to this setting. Thus $a, b \in A$ are adjacent if there does not exist $c \in A$ such that $(a, b, c) \in B$. As in the case of single-peaked preferences, $[a \sim b] \Rightarrow [a$ and b are adjacent$]$. An adjacency rich domain is a set of GSP orderings where the converse implication also holds. We let $\mathcal{D}^{GSP}$ denote the set of all GSP orderings.

The next two results and their proofs are straightforward adaptations of their counterparts for single-peaked domains.

Proposition **5** *Assume $N = 2$. Then every adjacency rich domain $\mathcal{D} \subset \mathcal{D}^{GSP}$ satisfies Property T and hence the tops-only property.*

Proposition **6** *Assume $N \geq 3$. The domain $\mathcal{D}^{GSP}$ satisfies Property T^* and hence the tops-only property.*

Observation **6** Nehring and Puppe (2007b) and Nehring and Puppe (2007a) demonstrate that a large number of models are covered by their abstract formulation of GSP preferences. These include single-peaked preferences on a tree (Demange (1982)) and multi-dimensional single-peakedness (Barberà et al. (1993)). Our results, Propositions 5 and 6, therefore cover these models as well.

4.5 Multi-dimensional Voting: Separable Preferences, Universal Marginal Domains

In this subsection we consider models where agents vote over alternatives which are multi-dimensional in nature, i.e the set of alternatives $A \equiv A_1 \times .... \times A_M$. The sets A_j, $j = 1, .., M$ will be referred to as component sets. The case where $A_j = \{0, 1\}$ was analyzed in Barberà et al. (1993) and the general case in LeBreton and Sen (1999). The literature surveyed in Sprumont (1995).

We shall write a typical element $a \in A$ as $a \equiv (a_1, ..., a_M)$ or (a_Q, a_{-Q}) where $Q \subset \{1, ..., M\}$. We shall consider separable preferences where it is possible to unambiguously define preferences over each component set.

Definition **22** *The ordering P_i is separable if for all $Q \subset \{1, ..., M\}$, and $a, b, c, d \in A$, $[(a_Q, c_{-Q})P_i(b_Q, c_{-Q}) \Rightarrow [(a_Q, d_{-Q})P_i(b_Q, d_{-Q})]$.*

We shall let $\mathcal{D}^{SEP}$ denote the set of all separable preferences over A. For every $P_i \in \mathcal{D}^{SEP}$, let P_i^j, $j = 1, ..., M$ denote the preference ordering on the j^{th} component set by P_i. We shall refer to P_i^j as the marginal ordering over j induced by P_i. For any $\mathcal{D} \subset \mathcal{D}^{SEP}$, let $\mathcal{D}^j = \{P_i^j | P_i \in \mathcal{D}\}$. Finally let $\mathcal{P}^j$ denote the set of all preference orderings over component j. Thus $\mathcal{D}^{SEPj} = \mathcal{P}^j$.

Definition **23** *The ordering P_i is additively representable if there exists functions $u_j : A_j \to \Re$, $j = 1, ..M$ such that for all $a, b \in A$, aP_ib if and only if $\sum_j u_j(a_j) > \sum_j u_j(b_j)$.*

Let $\mathcal{D}^{ADD}$ denote the set of all additively representable preferences. It is trivial to verify that $\mathcal{D}^{ADD} \subset \mathcal{D}^{SEP}$. It is also well-known that there exist separable preferences which are not additively representable. [4]

Proposition **7** *Let $\mathcal{D}$ be a domain such that $\mathcal{D}^{ADD} \subseteq \mathcal{D} \subseteq \mathcal{D}^{SEP}$. Then $\mathcal{D}$ satisfies the tops-only property.*

Proof: : We will show that a domain $\mathcal{D}$ such that $\mathcal{D}^{ADD} \subseteq \mathcal{D} \subseteq \mathcal{D}^{SEP}$ satisfies Property T^*.

Pick $P_i \in \mathcal{D}$, let $b = \tau(P_i)$ and $a \neq b$ and let $S = \{k \in \{1, .., M\} : a_k = b_k\}$. Let $H(P_i, a) = \{c \in A | c = (b_Q, a_{-Q})$ for some $Q \supset S\}$. [5] We claim that $\bar{B}(P_i, a) = H(P_i, a)$.

Let $P_i' \in \mathcal{D}^{SEP}$ be such that $\tau(P_i') = b$. Separability implies that whenever $S \subset Q$, we have $(b_Q, a_{-Q})P_i'(a_Q, a_{-Q})$. Hence $H(P_i, a) \subseteq \bar{B}(P_i, a)$. Now construct $u_j : A_j \to \Re$ as follows:

- $u_j(b_j) = 1$ for all $j = 1, ..M$.

[4] See for example, Fishburn (1970).

[5] In the definition of the set $H(P_i, a)$, we require Q to be a *strict* superset of S.

- $u_j(c_j) < \gamma$ for all $j \in S$ and for all $c_j \in A_j - \{b_j\}$ where $0 < \gamma < 1$.
- $u_j(a_j) = \beta_j$ and $u_j(c_j) < \gamma$ for all $j \in M - S$ and for all $c_j \in A_j - \{b_j, a_j\}$ where $0 < \gamma < \beta_j < 1$.

Let $\beta = \min_{j \in M-S} \beta_j$. Observe that $\sum_j u_j(a_j) \geq |Q| + \beta|M - Q| \geq \beta M$. Pick an arbitrary $c \notin H(P_i, a)$. There must exist $k \in \{1, .., M\}$ such that $c_k \neq a_k, b_k$. Therefore, $\sum_j u_j(c_j) \leq M - 1 + \gamma$. By picking β such that $\beta > \frac{M-1+\gamma}{M}$, we can ensure that $\sum_j u_j(a_j) > \sum_j u_j(c_j)$. Let $P'_i \in \mathcal{D}^{ADD}$ be the ordering generated by the functions u_j, $j = 1, .., M$. Observe that $\tau(P'_j) = b$ and $aP'_i c$. Hence $c \notin \bar{B}(P_i, a)$, i.e. $\bar{B}(P_i, a) = H(P_i, a)$.

Let $x \in H(P_i, a)$, i.e. $x = (a_S, b_{Q-S}, a_{-Q})$ for some $S \subset Q$. Now construct $u_j : A_j \to \Re$ as follows:

- $u_j(a_j) = 1$ for all $j = 1, ..M$.
- $u_j(c_j) < \gamma$ for all $j \in S \cup (M - Q)$ and for all $c_j \in A_j - \{a_j\}$ where $0 < \gamma < 1$.
- $u_j(b_j) = \beta_j$ and $u_j(c_j) < \gamma$ for all $j \in Q - S$ and for all $c_j \in A_j - \{b_j, a_j\}$ where $0 < \gamma < \beta_j < 1$.

Once again let $\beta = \min_{j \in Q-S} \beta_j$. Observe that $\sum_j u_j(x_j) \geq |M| + |Q - S| + \beta|Q - S| \geq \beta M$. Pick an arbitrary $c \notin H(P_i, a)$. There must exist $k \in \{1, .., M\}$ such that $c_k \neq a_k, b_k$. Therefore, $\sum_j u_j(c_j) \leq M - 1 + \gamma$. By picking β such that $\beta > \frac{M-1+\gamma}{M}$, we can ensure that $\sum_j u_j(x_j) > \sum_j u_j(c_j)$. Let $P'_i \in \mathcal{D}^{ADD}$ be the ordering generated by the functions u_j, $j = 1, .., M$. Observe that $\tau(P'_j) = a$ and $xP'_i c$ for all $c \notin \bar{B}(P_i, a)$ i.e. $x\bar{P}_i(A - \{a\} - \bar{B}(P_i, a))$. Since $\bar{B}(P_i, a) \subset B(P_i, a)$, we have $x\bar{P}_i(A - \{a\} - B(P_i, a))$ or $x\bar{P}_i W(P_i, a)$. Hence $\mathcal{D}$ satisfies Property T^*. We now apply Theorem 2. ■

4.6 Constrained Voting

A class of problems that has received considerable attention in the literature (Barberà et al. (1997) Barberà et al. (2005), Serizawa (1996), Aswal et al. (2003), Ozyurt and Sanver (2006) and Svensson and Torstensson (2008)) is that of constrained voting. There is a basic set of feasible alternatives A which has a product structure i.e., $A = A_1 \times A_2 \times ... \times A_M$ where A_j, $j = 1, ...M$ are the component sets. In Serizawa (1996) and Svensson and Torstensson (2008), the component set A_j denotes the level of public good j with the typical assumption that $|A_j| \geq 3$. On the other hand, Barberà et al. (1997) Barberà et al. (2005), Aswal et al. (2003), Ozyurt and Sanver (2008) consider models of voting over subsets of candidates. Thus $\{1, ..., M\}$ is the possible set of candidates and $A_j = \{0, 1\}$. Every $a \in A$, uniquely represents a subset of A with $a_j = 1$ and $a_j = 0$ signifying that j belongs and does not belong respectively to the subset represented by a.

The set of feasible alternatives is $B \subset A$. In the public goods case, these restrictions may reflect resource constraints (Serizawa (1996)) and in the voting over subsets of candidates, restrictions such as that at least k candidates must be elected, there must at least as many women elected as men Barberà et al. (2005) and so on.

The preferences of each voter is represented by a separable preference P_i over A. Let $\mathcal{D}^{SEP}$ denote the set of all separable preferences over A. For all $P_i \in \mathcal{D}^{SEP}$, let P_i^B denote the preferences over B induced by P_i, i.e. for all $a, b \in B$, $aP_i^B b$ if and only if $aP_i b$. As usual, we will let $r_k(P_i^B)$ and $\tau(P_i^B)$ denote respectively the kth ranked and first ranked alternatives in P_i^B. The set of preferences over B induced by separable preferences over A will be denoted by $\mathcal{D}^{SEP}(B)$.

Definition **24** *Let $\mathcal{D}(B) \subset \mathcal{D}^{SEP}(B)$. A constrained SCF f is a mapping $f : (\mathcal{D}(B))^n \rightarrow B$.*

Preferences over the set B are generated by separable preferences over the set A. We could alternatively define a constrained SCF f as a mappping $f : (\mathcal{D}^{SEP})^n \rightarrow B$. However, it is important to note that this distinction is irrelevant for strategy-proof constrained SCFs. Thus if f is strategy-proof and $P, \bar{P} \in (\mathcal{D}^{SEP})^n$ are such that $P_i^B = \bar{P}_i^B$ for all i, then $f(P) = f(\bar{P})$. This is an immediate consequence of the well-known fact that the value of a strategy-proof SCF f at a profile can depend only on individual preferences over alternatives in the range of f.

Definition **25** *The constrained SCF f satisfies the tops-only property if for all $P, \bar{P} \in \mathcal{D}(B)^n$ such that $\tau(P_i, B) = \tau(\bar{P}_i, B)$ for all $i \in N$, we have $f(P) = f(\bar{P})$.*

Observation **7** We are assuming that constrained SCFs f under consideration satisfy the unanimity property with respect to the feasible alternatives, i.e. for elements of the set B. An immediate consequence of this assumption is that $\text{Range } f = B$.

We will distinguish between two kinds of constrained voting models which give very different results. In one class of models, studied for instance in Barberà et al. (1997), each voters' unconstrained maximal element is assumed to be feasible. Thus $P_i^B \in \mathcal{D} \subset \mathcal{D}^{SEP}(B)$ implies that $\tau(P_i) \in B$ where $P_i \in \mathcal{D}^{SEP}$ induces P_i^B. For instance if $A = \{11, 10, 01, 00\}$, and $B = \{11, 10, 00\}$, then the ordering P_i^B where $11P_i^B 00P_i^B 10$ (or more compactly, $11, 00, 10$) is inadmissible because P_i^B can be induced only by the ordering P_i over A where $01, 11, 00, 10$ and $\tau(P_i) = 01 \notin B$. We will call this model, the *tops-feasible* model. The other model where no restrictions are made on the feasibility of the top alternatives , will be referred to as the *unrestricted-tops* model. This model has been studied in Barberà et al. (1997), Aswal et al. (2003), Ozyurt and Sanver (2008) and Svensson and Torstensson (2008).

The structure of strategy-proof SCFs in these two models can be quite different. In the feasible-tops model, we show that Property T^* holds. In the unrestricted-tops model, we provide an example to show that Property T^* does not hold. Since the domain in the example can be shown to be dictatorial, the example demonstrates that Property T^* is not necessary for tops-onlyness.

4.6.1 The Tops-Feasible Model

As indicated earlier, we assume that preferences over the feasible set B are generated by separable preferences over the set B subject to the restriction that the peaks of all admissible orderings are feasible. The domain that we will examine is the largest one that can be generated with these restrictions. For instance, consider the example introduced earlier where $A = \{11, 10, 01, 00\}$ and $B = \{11, 10, 00\}$. The domain we will consider consists of the following orderings

$$\{(11, 10, 00), (10, 11, 00), (10, 00, 11), (00, 10, 11)\}.$$

Generally,

$$\bar{\mathcal{D}}(B) = \{P_i^B \in \mathcal{D}^{SEP}(B) | \tau(P_i^B) \in B\}$$

Proposition 8 *The domain $\bar{\mathcal{D}}(B)$ has the tops-only property.*

Proof: : We will show that $\bar{\mathcal{D}}(B)$ satisfies Property T^*.

Let $P_i^B \in \bar{\mathcal{D}}(B)$, $\tau(P_i^B) = b \in B$ and $a \in B$ with $a \neq b$. Let $P_i \in \mathcal{D}^{SEP}$ induce P_i^B. [6] Clearly $B(P_i^B, a) \subseteq B(P_i, a)$. Hence $\bar{B}(P_i^B, a) \subseteq \bar{B}(P_i, a)$. Now pick $x \in \bar{B}(P_i^B, a)$. Since $x \in \bar{B}(P_i, a)$, we can apply Proposition 7 to conclude that there exists $\bar{P}_i \in \mathcal{D}^{SEP}$ such that $\tau(\bar{P}_i) = a$ and $x\bar{P}_i W(P_i, a)$. Let $\bar{P}_i$ induce $\bar{P}_i^B$. Obviously $\tau(\bar{P}_i^B) = a$. Also since $W(\bar{P}_i^B, a) \subset W(\bar{P}_i, a)$ we have $xP_i^B W(P_i^B, a)$. Hence $\bar{\mathcal{D}}(B)$ satisfies Property T^*. ■

Observation 8 In the example where $A = \{11, 10, 01, 00\}$ and $B = \{11, 10, 00\}$ the domain $\bar{\mathcal{D}}(B)$ is single-peaked with respect to the ordering $>$ where $11 > 10 > 00$. [7] There is therefore a rich class of non-dictatorial strategy-proof SCFs over this domain such as the median voter rule. However all these strategy-proof SCFs satisfy tops-onlyness.

[6] Of course, P_i need not be unique.

[7] Also with respect to the ordering $>$ where $00 > 10 > 11$.

4.6.2 The Unrestricted-Tops Model

We now consider the case where the peaks of admissible preferences need not be feasible. This case is harder to deal with than the unrestricted-tops case. We show that $\mathcal{D}^{SEP}(B)$ may violate Property T^*.

Example **5** Let $A = A_1 \times A_2 \times A_3$ where $A = \{0,1\} \times \{0,1\} \times \{0,1\}$. Let $B = A - \{011\}$. Let P_i be the ordering

$$((011),(111),(001),(010),(000),(101),(110),(100))$$

It can be verified that $P_i \in \mathcal{D}^{SEP}$. Clearly $\tau(P_i^B) = (111)$. Let $a = (000)$. Note that for any $P_i' \in \mathcal{D}^{SEP}$ such that either $\tau(P_i') = (111)$ or $\tau(P_i') = (011)$ and $r_2(P_i') = (111)$, we must have $(001)P_i'(000)$ and $(010)P_i'(000)$. Hence $\bar{B}(P_i^B, a) = \{111, 010, 001\}$. If Condition T^* is to be satisfied, there must exist $\bar{P}_i^B \in \mathcal{D}^{SEP}$ such that $\tau(\bar{P}_i^B) = (000)$ and $(111)\bar{P}_i^B W(P_i,(000))$. Suppose that such a $\bar{P}_i^B$ exists. There are two cases to consider. Suppose first that $\tau(\bar{P}_i) = (000)$. Then (111) is ranked last in $\bar{P}_i$. Hence $(101)\bar{P}_i^B(111)$ so that the requirement that $(111)\bar{P}_i^B W(P_i,(000))$ is violated. Suppose that $\tau(\bar{P}_i) \notin B$, i.e $\tau(\bar{P}_i) = (011)$. Then separability of $\bar{P}_i$ implies $(010)\bar{P}_i(000)$ so that the requirement that $\tau(\bar{P}_i^B) = (000)$ is violated.

The results in Aswal et al. (2003) can be used to show that domain in Example 5 is dictatorial. Therefore the example demonstrates that Property T^* is not necessary for tops-onlyness.

5 Non-Minimally Rich Domains

In this section we relax the minimal richness assumption. There are several well-known domains where minimal richness fails, for instance, in models where sets are being ranked according to an expected utility criterion (see Barberà et al. (2001)).

The general results obtained earlier cannot be easily adapted to this setting. To begin with, the notion of connectedness between two alternatives is now severely restricted because it cannot be applied to alternatives which are not ranked first in some admissible ordering. Consequently, Property T can never be satisfied in these models so that Theorem 1 can never be applied. Can Property T^* be generalized in order to cover alternatives which are in the range of the SCF but which are never first-ranked in any admissible ordering? This appears to be difficult without reference to the specific structure of option sets. Suppose $a \in O_i(P_{-i})$ for some admissible $n-1$ profile P_{-i} and let $(P_j', P_{-i,j})$ be another admissible $n-1$ profile which is tops equivalent to P_{-i}. Steps 1 and 2 in the proof of Theorem 2 go through as before so that we can conclude that if $a \in O_i(P_{-i}) - O_i((P_j', P_{-i,j})$, then there exists $x^* \in O_i(P_{-i}) \cap O_1((P'_j, P_{-i}))$ where $x^* \in \bar{B}(P_j, a)$. Property T^* requires the existence of an ordering where a is ranked first and x^* is preferred to all alternatives that a was preferred to under P_j. Since $a \in O_i(P_{-i})$, we know that there exists an admissible P_i such that $f(P_i, P_{-i}) = a$. In this ordering P_i, the first ranked alternative, say b clearly does not belong to $O_i(P_{-i})$. So, if we could postulate the existence of an ordering P_i where b is the first ranked, a is the highest ranked amongst all alternatives in $O_i(P_{-i})$ and x^* is preferred to all alternatives that a was better than under P_j, then we would have the required counterpart to Property T^*. Unfortunately, the construction of such an ordering requires knowledge of $O_i(P_{-i})$ (for instance, how do we identify b?) which renders the entire approach unworkable.

In view of these difficulties, we proceed as follows. We show that partial results relating to the properties of alternatives which can be ranked first, do carry over. Moreover, using these partial results, general tops-onlyness results can be obtained in specific models using the appropriate structure of option sets.

Fix a domain $\mathcal{D}$. Let $A_0 \subset A$ denote the set of alternatives with the property that for all $a \in A_0$ there exists $P_i \in \mathcal{D}$ such that $\tau(P_i) = a$. Clearly $A_0 \neq \emptyset$ but it could be the case that A_0 is a strict subset of A. We shall assume that SCFs under consideration satisfy *unanimity* which is defined exactly as before. The definition of tops-onlyness also carries over without any changes or qualifications. The following definitions are however, specific to this environment.

Definition **26** *The SCF $f : \mathcal{D}^n \to A$ satisfies partial tops-onlyness if, for all tops-equivalent profiles $P, P' \in \mathcal{D}^n$, $f(P) \in A_0 \Rightarrow [f(P) = f(P')]$. The domain $\mathcal{D}$ satisfies partial tops-onlyness if every strategy-proof and unanimous SCF $f : \mathcal{D}^n \to A$ satisfies partial tops-onlyness.*

In other words, a SCF is partially tops-only if it satisfies tops-onlyness with respect to the alternatives in A_0. A domain satisfies partial tops-onlyness if every strategy-proof and unanimous SCF defined over this domain satisfies partial tops-onlyness. We will show that Property T^* can be modified in a natural way so that Theorem 2 can be replicated in order to obtain a result on partial tops-onlyness. We will then show that in a variety of specific domains, the partial tops-onlyness result can be extended to obtain results on tops-onlyness.

As before, for all $P_i \in \mathcal{D}$ and $a \in A - \tau(P_i)$, let $B(P_i, a) = \{x \in A | xP_ia\}$ and $W(P_i, a) = \{x \in A | aP_ix\}$. Also let $\bar{B}(P_i, a) = \{x \in A | x \in B(P'_i, a)$ for all P'_i such that $\tau(P_i) = \tau(P'_i)\}$.

Definition **27** *Fix a domain $\mathcal{D}$. Let $a \in A_0$ and $P_i \in \mathcal{D}$ be such that $a \neq \tau(P_i)$. Then a is connected for P_i if for all $x \in \bar{B}(P_i, a)$ there exists $\bar{P}_i$ such that $\tau(\bar{P}_i) = a$ and $x\bar{P}_iW(P_i, a)$. Moreover $\mathcal{D}$ satisfies Property NRT^* if for all $P_i \in \mathcal{D}$ and $a \in A_0 - \tau(P_i)$, a is connected for P_i.*

The requirement for an alternative to be connected in Definition 27 is exactly the same as that in Definition 9. However Property NRT^* only requires alternatives in A_0 to be

P_1	P_2
a_1	a_1
a_{12}	a_{13}
a_2	a_{12}
a_{23}	a_2
a_{13}	a_{23}
a_3	a_3

Table 3: Examples of In-Between Preferences

connected. There is a reason for this: observe that alternatives in $A - A_0$ cannot be connected because the definition of connection requires the existence of an admissible ordering where these orderings are ranked first. Observe that in terms of option sets, a SCF f satisfies partial tops-onlyness if for $P_{-i}, P'_{-i} \in \mathcal{D}^{n-1}$, we have $O_i(P_{-i}) \cap A_0 = O_i(P'_{-i}) \cap A_0$ whenever P_{-i} and P'_{-i} are tops-equivalent.

THEOREM **5** *Let $\mathcal{D}$ be a domain satisfying Property NRT^*. Then, $\mathcal{D}$ satisfies partial tops-onlyness.*

The proof of Theorem 5 is straightforward adaptation of the proof of Theorem 2 and is omitted. In the examples below we show that Theorem 5 can be used to prove tops-onlyness.

5.1 In-Between Preferences

In this model, the set of alternatives A consists of *singletons* and *pairs*. There are m singletons $\{a_1, .., a_m\}$ and $\frac{m(m-1)}{2}$ pairs, $\{a_{jk}\}$ where $j, k \in \{1, ..., m\}$. A singleton a_j can be thought of as a "pure" outcome and a pair a_{jk} as a "compromise" between singletons a_j and a_k. One possible interpretation (though not the only one) is that a compromise a_{jk} is an even-chance lottery between a_j and a_k.

We will assume that admissible preferences P_i satisfy the following restriction which we call the "in-betweenness" restriction: for all $j, k \in \{1, .., m\}$ with $j \neq k$, either $a_j P_i a_{jk} P_i a_k$ holds or $a_k P_i a_{jk} P_i a_j$ holds. In other words the compromise a_{jk} must lie "in-between" the pure outcomes a_j and a_k. This will be satisfied if a compromise is an even-chance lottery and orderings are based on an expected utility calculation. More generally, it will be satisfied if preferences satisfy the axiom of "Averaging" discussed in Fishburn (1970) and Gravel et al. (2008). An immediate consequence of in-betweenness is that only singletons can be ranked first, i.e. $A_0 = \{a_1, a_2, .., a_m\}$. For convenience, we denote the set of pairs by A_1 so that $A = A_0 \cup A_1$. In Table 3, P_1 and P_2 are examples of orderings satisfying in-betweenness when $A_0 = \{a_1, a_2, a_3\}$.

Let $\mathcal{D}^{IB}$ denote the set of all orderings satisfying in-betweenness. We show that this domain satisfies tops-onlyness. We show this by first showing that it satisfies partial tops-onlyness by virtue of satisfying Property NRT^* and then extending the result.

PROPOSITION **9** *The domain $\mathcal{D}^{IB}$ satisfies the tops-onlyness property.*

Proof: Our first observation is that if $|A_0| = 2$, then there is a unique ordering associated with every singleton so that tops-onlyness is trivially satisfied. Assume therefore that $|A_0| \geq 3$. We will first show that $\mathcal{D}^{IB}$ satisfies the partial tops-onlyness property by showing that it satisfies Property NRT^* and applying Theorem 5. We then use this result to demonstrate tops-onlyness.

Let $P_i \in \mathcal{D}^{IB}$, $a_j = \tau(P_i)$ and $a_k \in A_0 - \{a_j\}$. Since there exists an in-between preference ordering where a_j, a_{jk} and a_k are ranked first, second and third respectively, it follows $\bar{B}(P_i, a_k) \subset \{a_j, a_{jk}\}$. Since every ordering satisfying in-betweenness with a_j as the maximal alternative must rank a_{jk} above a_k, it follows that $\{a_j, a_{jk}\} = \bar{B}(P_i, a_k)$. Let $\bar{P}_i \in \mathcal{D}^{IB}$ be such that a_k, a_{jk} and a_j are ranked first, second and third respectively. Since $x\bar{P}_i W(P_i, a_k)$ for all $x \in \bar{B}(P_i, a_k)$, Property NRT^* is satisfied and partial tops-onlyness follows from Theorem 5.

Let $f : (\mathcal{D}^{IB})^n \to A$ be a strategy-proof satisfying unanimity. Pick $i \in N$. We will now show that $O_i(P_{-i}) = O_i(P'_{-i})$ whenever P_{-i} and P'_{-i} are tops-equivalent. Note that this implies that f is tops-only and moreover the proof of this does not require minimal richness (see Proposition 1). We shall closely follow the arguments in the proof of Theorem 2. Let $a_{j,k} \in O_i(P_{-i})$, $t \in N - \{i\}$ and suppose $\tau(P_t) = a_t$. Let P'_t be such that $\tau(P'_t) = a_t$ but $a_{jk} \notin O_i((P'_t, P_{-\{i,t\}}))$. Following Steps 1 and 2 in the proof of Theorem 2 which make no use of either minimal richness or Property T^*, we conclude that there exists $x^* \in \bar{B}(P_t, a_{jk})$ which belongs to both $O_i((P_t, P_{-\{i,t\}}))$ and $O_i((P'_t, P_{-\{i,t\}}))$. By the in-betweenness restriction, exactly one of $a_j P_t a_{jk} P_t a_k$ or $a_k P_t a_{jk} a_j$ must hold. Assume without loss of generality that the former holds. Using the same argument as in the previous paragraph, it is easy to verify that $\bar{B}(P_t, a_{jk}) = \{a_t, a_{jt}, a_j\}$.

Since $a_{jk} \in O_i((P_t, P_{-\{i,t\}}))$, there must exist $P_i \in \mathcal{D}^{IB}$ such that $f(P_i, P_t, P_{-\{i,t\}}) = a_{jk}$. Suppose $a_j P_i a_{jk} P_i a_k$. Let $P'_i \in \mathcal{D}^{IB}$ be such that a_j, a_{jk}, a_{jt} and a_t are ranked first, second, third and fourth respectively. Since f is strategy-proof, it follows from standard arguments that $f(P'_i, P_t, P_{-\{i,t\}}) = a_{jk}$ and in addition, that $a_j \notin O_i(P_t, P_{-\{i,t\}})$. Since f satisfies partial tops-onlyness, $a_j \notin O_i(P'_t, P_{-\{i,t\}})$. We know the following: (i) $a_{jk} \notin O_i(P'_t, P_{-\{i,t\}})$ by hypothesis (ii) there exists $x^* \in \{a_t, a_{jt}, a_j\}$ such that $x^* \in O_i((P'_t, P_{\{i,t\}}))$. Observe that (i) and (ii) imply $f(P'_i, P'_t, P_{-\{i,t\}}) \in \{a_t, a_{j,t}\}$. Since both a_t and a_{jt} are preferred by t to $a_{j,k}$ under P_t , it follows that t will manipulate at $(P'_i, P_t, P_{-\{i,t\}})$ via P'_t.

Now suppose $a_k P_i a_{jk} P_i a_j$ but $a_{jk} \notin O_i((P'_t, P_{-\{i,t\}}))$. Let $P''_i \in \mathcal{D}^{IB}$ be such that a_k, a_{jk}, a_{jt} and a_t are ranked first, second third and fourth respectively. Once again,

$f(P_i'', P_t, P_{-\{i,t\}}) = a_{jk}$ and $a_k \notin O_i(P_t, P_{-\{i,t\}})$. From the partial tops-onlyness of f, we have $a_k \notin O_i(P_t', P_{-\{i,t\}})$. Hence $f(P_i'', P_t', P_{-\{i,t\}}) \in \{a_t, a_{jt}, a_j\}$. Since t prefers a_t, a_{jt} and a_j to a_{jk} under P_t, it follows that she will manipulate at $(P_i'', P_t, P_{-\{i,t\}})$ via P_t'. Therefore $O_i(P_t, P_{-\{i,t\}})) = O_i(P_t', P_{-\{i,t\}}))$. Applying this argument repeatedly for different voters, we conclude that $O_i(P_{-i}) = O_i(P_{-i}')$ and f satisfies tops-onlyness. ■

Observation **9** The tops-onlyness property can be used to show quite easily that $\mathcal{D}^{IB}$ is in fact, dictatorial if $|A_0| \geq 3$. We merely outline the argument here. We first show that a strategy-proof f satisfying unanimity cannot have any pairs in its range. Suppose this was false and there existed a profile P and a pair a_{jk} such that $f(P) = a_{jk}$. By the tops-onlyness property, we can assume without loss of generality that a_{jk} is in the fact, the second ranked alternative from the bottom for all voters. Now pick a_r distinct from a_j and a_k and let P' be a profile where every voter's first-ranked alternative is a_r, and the position of a_{jk} and the bottom ranked alternative remain the same as in P. A standard argument can be used to show that $f(P') = a_{jk}$. However unanimity requires $f(P') = a_r$. Hence Range $f = A_0$. Since the value of a strategy-proof SCF depends only on voters' rankings over the range of the SCF, it follows that f can depend only on rankings over A_0. Since these rankings are unrestricted, the Gibbard-Satterthwaite Theorem can be applied to show that f is dictatorial.

5.2 Kelly's domain

Consider the following domain denoted by $\mathcal{D}^K$. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. Then $\mathcal{D}^K$ is the largest domain of orderings satisfying the restrictions below.

- a_4 lies between a_1 and a_3
- a_5 lies between a_1 and a_3
- a_6 lies between a_2 and a_3
- a_7 is neither first nor last.

Kelly (1989) conjectured that there does not exist a strategy-proof Pareto-efficient, non-dictatorial SCF defined over $\mathcal{D}^K$ with range including the alternatives, a_1, a_2 and a_3. Kim and Roush (1989) proved a stronger version of this conjecture showing that the requirement of Pareto-efficiency is redundant. Here we show that the arguments in the proof of Proposition 9 extend easily to this case.

Proposition **10** *The domain $\mathcal{D}^K$ satisfies the tops-only property.*

Proof: In this case $A_0 = \{a_1, a_2, a_3\}$. Let f be a strategy-proof SCF with $\{a_1, a_2, a_3\} \subseteq$ Range f. It follows from well-known arguments that f satisfies unanimity.

Identifying the outcomes a_1, a_2 and a_3 as singletons and the alternatives a_4 and a_5 and a_6 as pairs, the arguments in the proof of Proposition 9 extend in a straightforward manner to enable us to conclude that f satisfies top-onlyness with respect to the alternatives, a_1, a_2, a_3, a_4, a_5 and a_6. In addition the arguments in Observation 9 also extend in identical fashion to allow us to conclude that $a_4, a_5, a_6 \notin$ Range f.

We now show that the tops-only property also holds with respect to a_7. Suppose that it does not hold, i.e. there exists $i, t \in N$ and tops-equivalent profiles $P_{-i}, (P'_t, P_{-\{i,t\}}) \in (\mathcal{D}^K)^{n-1}$ such that $a_7 \in O_i(P_{-i}) - O_i((P'_t, P_{-\{i,t\}}))$. Once again, applying Steps 1 and 2 in the proof of Theorem 2, it follows that there exists $x^* \in O_i(P_{-i}) \cap O_i((P'_t, P_{-\{i,t\}}))$ and $x^* \in \bar{B}(P_t, a_7)$. Assume without loss of generality that $\tau(P_t) = a_1$. It is easy to check that $x^* = a_1$. Since $a_7 \in O_i(P_{-i})$, there exists $P_i \in \mathcal{D}^K$, such that $f(P_i, P_{-i}) = a_7$. Notice that $\tau(P_i) \neq a_1$ because $a_1 \in O_i(P_{-i})$. Assume $\tau(P_i) = a_j$ where $j = \{2, 3\}$. Observe also that $a_j \notin O_i(P_{-i})$ (because $f(P_i, P_{-i}) = a_7$). By partial tops-onlyness $a_j \notin O_i((P'_t, P_{-\{i,t\}})$.

Moreover we can assume without loss of generality that a_7 and a_1 are ranked second and third respectively amongst the alternatives $\{a_1, a_2, a_3, a_7\}$ in P_i. Then $f(P'_t, P_i, P_{-\{i,t\}}) = a_1$ (since the alternatives a_4, a_5 and a_6 are not in Range f). Since $a_1 P_t a_7$, voter t manipulates at $(P_t, P_i, P_{-\{i,t\}})$ via P'_t. Hence $O_i(P_{-i}) = O_i((P'_t, P_{-\{i,t\}})$ and f satisfies the tops-only property. ■

5.3 Single-Dipped Preferences

These domains have been studied in, for instance Peremans and Storcken (1997) and Klaus et al. (1997). It will be convenient to represent the set of alternatives by $A = \{a_1, ..., a_M\}$. Let $>$ be the linear order $a_m > a_{M-1} ... > a_1$.

Definition **28** *The ordering P_i is said to be Single-Dipped (with respect to $>$)if there exists an alternative* $\min(P_i)$ *such that for all $a, b \in A$,* $[\min(P_i) > b > a$ *or* $a > b > \min(P_i)] \Rightarrow$ *aP_ib.*

Thus the alternative $\min(P_i)$ is the worst element according to P_i and the further an alternative is from $\min(P_i)$, the better it is. Let $\mathcal{D}^{SD}$ denote the set of all single-dipped preferences with respect to P_i.

Proposition **11** *The domain $\mathcal{D}^{SD}$ satisfies partial tops-onlyness.*

Proof: : Observe that $A_0 = \{a_1, a_m\}$. We first show that $\mathcal{D}^{SD}$ satisfies partial tops-onlyness. Let $P_i \in \mathcal{D}^{SD}$ and assume without loss of generality that $\tau(P_i) = a_1$. Since there exists a single-dipped ordering where a_1 and a_m are ranked first and second respectively, it follows that $\bar{B}(P_i, a_m) = \{a_1\}$. Since there exists a single-dipped ordering where a_m is first and a_1 second, Property NRT^* is satisfied and $\mathcal{D}^{SD}$ satisfies partial tops-onlyness. ■

Observation **10** It is possible to show tops-onlyness in this model but we omit the details.

5.4 Preferences over sets derived from the expected utility hypothesis

We consider the model of ranking all subsets of a finite set studied in Barberà et al. (2001), Benoit (2002), Ching and L.Zhou (2002), Duggan and Schwartz (2000) and Ozyurt and Sanver (2006). Let $A_0 = \{a_1, .., a_m\}$ be the set of all singletons and let A denote the set of all non-empty subsets of A_0. Barberà et al. (2001) introduce the following restrictions on preference orderings over A.

Definition **29** *An assessment λ is a function $\lambda : A_0 \to [0, 1]$ such that $\lambda(a_j) > 0$ for all $a_j > 0$ and $\sum_{a_j \in A_0} \lambda(a_j) = 1$.*

Definition **30** *The ordering P_i is conditionally expected utility consistent (CEUC) if there exists a utility function $v_i : A_0 \to \Re$ and an assessment λ_i such that, for all $X, Y \in A$*

$$XP_iY \Leftrightarrow \sum_{a_j \in X} v_i(a_j)\left(\frac{\lambda_i(a_j)}{\sum_{a_k \in X} \lambda_i(a_k)}\right) > \sum_{a_j \in Y} v_i(a_j)\left(\frac{\lambda_i(a_j)}{\sum_{a_k \in Y} \lambda_i(a_k)}\right)$$

Let $\mathcal{D}^{CEUC}$ denote the set of all conditionally expected utility consistent utility orderings. A closely related domain is the domain of all uniform expected utility consistent preferences $\mathcal{D}^{UEU}$ preferences.

Definition **31** *The ordering P_i is uniform expected utility consistent (UEU) if there exists a utility function $v_i : A_0 \to \Re$ such that, for all $X, Y \in A$*

$$XP_iY \Leftrightarrow \sum_{a_j \in X} v_i(a_j)\left(\frac{1}{|X|}\right) > \sum_{a_j \in Y} v_i(a_j)\left(\frac{1}{|Y|}\right)$$

Barberà et al. (2001) characterize strategy-proof SCFs over these domains and Ozyurt and Sanver (2006) refine and extend these results. In particular, Barberà et al. (2001) show that the domain $\mathcal{D}^{CEUC}$ is dictatorial while $\mathcal{D}^{UEU}$ is *bi-dictatorial*, i.e. both domains satisfy the tops-only property. We show below that it is easy to verify that the domains satisfy Property NRT^* and therefore satisfy partial tops-onlyness. Unlike the previous three examples, it is considerably more tedious to verify that tops-onlyness is satisfied. We therefore do not undertake to demonstrate the more general tops-only property.

Proposition **12** *The domains $\mathcal{D}^{CEUC}$ and $\mathcal{D}^{UEU}$ satisfy partial tops-onlyness.*

Proof: Let P_i belong to either $\mathcal{D}^{CEUC}$ or $\mathcal{D}^{UEU}$ and suppose $\tau(P_i) = a_j$. Pick $a_k \neq a_j$. Note that by choosing a utility function v_i where $v_j(a_j) = 1$ and $v_j(a_k)$ is very close to 1, we can generate a P_i' which belongs to both $\mathcal{D}^{CEUC}$ and $\mathcal{D}^{UEU}$ such that a_j, $\{a_j, a_k\}$ and a_k are ranked first, second and third respectively. Hence $\bar{B}(P_i, a_j) = \{a_j, \{a_j, a_k\}\}$. By an analogous argument, we can find $\bar{P}_i$ belonging to both $\mathcal{D}^{CEUC}$ and $\mathcal{D}^{UEU}$ where a_k, $\{a_j, a_k\}$ and a_j are ranked first, second and third respectively. Thus $x^* \bar{P}_i W(P_i, a_k)$ for all $x^* \in \bar{P}(P_i, a)$. Thus, Property NRT^* is satisfied and $\mathcal{D}^{CEUC}$ and $\mathcal{D}^{UEU}$ both satisfy partial tops-onlyness. ■

6 Conclusion

We have investigated the structure of domains which imply the tops-onlyness property. A general characterization of this property poses difficulties which appear to be insurmountable. We provide some sufficient conditions and are able to show that they apply widely.

Several questions remain. The most obvious is whether Property T is sufficient in the general case. Is it possible to construct a domain which is tops-only for two voters and not for more than three?

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