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Semiparametric Estimation of Signaling Games

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Abstract

This paper studies an econometric modeling of a signaling game with two players where one player has one of two types. In particular, we develop an estimation strategy that identifies the payoffs structure and the distribution of types from data of observed actions. We can achieve uniqueness of equilibrium using a refinement, which enables us to identify the parameters of interest. In the game, we consider non-strategic public signals about the types. Because the mixing distribution of these signals is nonparametrically specified, we propose to estimate the model using a sieve conditional MLE. We achieve the consistency and the asymptotic normality of the structural parameters estimates. As an alternative, we allow for the possibility of multiple equilibria, without using an equilibrium selection rule. As a consequence, we adopt a set inference allowing for multiplicity of equilibria.

Keywords: Semiparametric Estimation, Signaling Game, Set Inference, Infinite Dimensional Parameters, Sieve Simultaneous Conditional MLE

JEL Classification: C13, C14, C35, C62, C73

1 Introduction

The econometric modeling of game theories has been of significant interest over the last decade. It has been one of most vivid research topics in the field of empirical industrial organization including studies on industry entry decisions (Bresnahan and Reiss (1990, 1991), Berry (1992), Toivanen and Waterson (2000), Seim (2002), Ciliberto and Tamer (2003)) and others (Mazzeo (2002), Sweeting (2004), Davis (2005)). Most of econometric analysis of game theoretic models has been focused on simultaneous games with complete information (Bjorn and Vuong (1984, 1985), Bresnahan and Reiss (1990, 1991), Tamer (2003), Bajari, Hong and Ryan (2004)) or with incomplete information (Brock and Durlauf (2001, 2003), Seim (2002), Sweeting (2004), Aradillas-Lopez (2005)). More recently, there are also many studies on dynamic games (Aguirregabiria and Mira (2003), Bajari, Benkard, and Levin (2003), Berry, Ovstrovsky, and Pakes (2003), Pesendorfer and Schmidt-Dengler (2003)).

However, although the information asymmetry problem has been noted widely both theoretically or empirically (Riley (2001) provides a valuable survey on this issue), a formal econometric analysis of a signaling

game appears only in few studies[1] for several reasons[2]. What has hindered the econometric implementation of these games is the presence of multiple equilibria (or even absence of any equilibrium), which results in the nonexistence of a well-defined likelihood function over the entire set of observable outcomes. Even when these problems are overcome, it is generally hard to set up a flexible econometric model that is able to deal with a generic family of signaling games. This paper studies the identification and the estimation of a signaling game with two players where one player, *informed player*, has one of two types. In particular, we develop an estimation strategy that identifies the payoffs structure and the distribution of types from data of observed actions by the two players. Though multiple equilibria arise in the game we study, an equilibrium refinement such as *intuitive criterion* by Cho and Kreps (1987) enables us to obtain the uniqueness of equilibrium.

Providing more empirical relevance to the basic model, we introduce some non-strategic public signals about the types, which the *informed player* can not manipulate technically, or at least has no incentive to do so. These signals are observed by all players and econometricians. Under the separating equilibrium, the action of the *informed player* reveals the true type and thus the *uninformed party* has no incentive to use the additional informations. However, under the pooling equilibrium, the *uninformed party* will update her belief on types using these signals following the Bayes' rule. We specify the mixing distribution of non-strategic signals on types nonparametrically and thus estimate the model using a sieve conditional MLE (a conditional MLE version of Wong and Severini (1991) or Shen (1997)) where the infinite dimensional parameters are approximated by sieves. Noting that the conditional probability of choosing a certain combination of actions can be written in terms of several conditional moment restrictions, as an alternative, we can also estimate the model using the sieve minimum distance (MD) estimation developed by Ai and Chen (2003). In both methods, we can obtain the consistency and the asymptotic normality of structural parameters estimates.[3]

As noted earlier, in the signaling game, the existence of multiple equilibria naturally arises given some realization of payoffs. We have resolved the multiple equilibria issue using a refinement of equilibrium (which is an equilibrium selection rule) as common in the literature. Even though players may select one equilibrium out of multiple equilibria using a selection rule, it is often hard to model or justify such a selection mechanism. Examples of papers that handle the multiple equilibria problem in other games under some strong assumption on the equilibrium selection rules include Bjorn and Vuong (1984, 1985), Kooreman (1994), and Bajari, Hong and Ryan (2004) in games of complete information. Another example is Sweeting (2004) in a game with incomplete information. The disadvantage of this approach is that there is no generally accepted procedure to determine which equilibrium will be played among multiple equilibria even though an equilibrium selection of a signaling game is more acceptable than that of a simultaneous game, at least theoretically (See Banks and Sobel (1987) and Cho and Kreps (1987)). Other options dealing with multiple equilibria include the redefinition of a game in a way that makes it estimable without requiring an equilibrium selection rule. Some studies redefine the space of outcomes of the game so that it exhibits uniqueness of equilibrium (Bresnahan and Reiss (1990, 1991)). These approaches have the merit that we do not need additional assumptions to justify the equilibrium selection but it is noted that such redefinitions may result in some loss of information in the game.

Inspired by important work by Manski and co-authors (Manski (1990), Horowitz and Manski (1995), Manski and Tamer (2002)) on bound analysis, some researchers have started to develop set inferences rather than a point estimation, without attempting to resolve the equilibrium selection (Sutton (2000), Ciliberto and Tamer (2003), Andrews, Berry, Jia (2004)). We note that for signaling games, the *self-fulfilling property* is essential, which yields multiple equilibria depending on different beliefs on plays of the other party. Thus, as

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[1]Ackerberg (2003) provides a dynamic learning model of consumer behavior allowing for the signaling effect of advertising on experience goods. Brown (2002) provides a structural estimation of prestige effects in IPO underwriting within a signaling game context where a firm's owner has an incentive to signal the firm's type by choosing prestigious underwriters.

[2]We admit that there exist many studies that empirically test testible implications of a particular signaling game but there are very few studies that actually estimate the game's primitives.

[3]This paper only presents a large sample theory for the sieve conditional MLE. The original dissertation contains the asymptotic results for both methods.

an attractive alternative to the previous approach, we allow for the possibility of multiple equilibria, without attempting to resolve the equilibrium selection problem. In particular, we consider the model where some asymptotic inequalities may define a region of parameters rather than a single point in the parameter space. By definition, when there are multiple equilibria, there exist regions of unobservables that are consistent with the necessary conditions for more than one equilibrium. Therefore, the probability implied by the necessary condition for a given event is greater than or equal to the true probability of the event. Some necessary conditions for a perfect Bayesian equilibrium (PBE) provide a set of inequality constraints on the parameters for the games we study. We illustrate that we can allow for the multiplicity of equilibria using Andrews, Berry, and Jia (2004)'s results.

The structure of this paper is as follows. Section 2 describes the game we study. In Section 3, we characterize equilibria of the game. In Section 4, we consider public information about the distribution of types. In Section 5, we estimate the model using a sieve conditional MLE and a sieve MD. The consistency and the asymptotic normality of the structural parameters estimates are presented. Section 6 briefly considers the set inference of the model. We conclude in Section 7. Some technical details and mathematical proofs are presented in the Appendix.

2 Description of the Game

To introduce the game of our interest, we first define a *Bayesian extensive game with observable actions*. The game models a situation where each player observes the actions of all players including herself but has uncertainty about payoffs that are affected by types of other players where the type of each player is informed only to that player not to the other players. We define this game formally following Osborne and Rubinstein (1994). First, we say that *chance* or *nature* selects types for the players and refer to player i after she receives the information t_i as type t_i.

Definition 2.1 *A Bayesian extensive game with observable actions is a tuple* $\langle \Gamma, (\Xi_i), (p_i), (u_i) \rangle$ *where*

- $\Gamma = \langle N, H, P \rangle$ *is an extensive game form with perfect information and simultaneous moves where N denotes the set of players; H is a history which collects actions taken by players. The set of terminal histories is denoted by $\boldsymbol{T}$; P is a player function that assigns a player who takes an action after the history $h \in H$.*
- Ξ_i *is a finite set of possible types of player i; we write $\Xi = \times_{i \in N} \Xi_i$.*
- p_i *is a probability measure on Ξ_i for which $p_i(t_i) > 0$ for all $t_i \in \Xi_i$, and the measures p_i are stochastically independent across i ($p_i(t_i)$ is the probability that player i is selected to be of type t_i).*
- $u_i : \Xi \times \boldsymbol{T} \to \mathbb{R}$ *is a von Neumann-Morgenstern utility function ($u_i(t, h)$ is player i's payoff when the profile of types is t and the terminal history of Γ is h).*

Now we define a *signaling game* as

Definition 2.2 *A signaling game is a Bayesian extensive game with observable actions* $\langle \Gamma, (\Xi_i), (p_i), (u_i) \rangle$ *in which Γ is a two-player game form where Player 1 takes an action then Player 2 takes an action, the set of Player 2's type (Ξ_2) is a singleton.*

What makes a signaling game interesting is that Player 2 (*uninformed party* or *receiver*) controls the action and Player 1 (*informed party* or *sender*) controls the information. The receiver has an incentive to try to deduce the sender's type from the sender's message (signal), and the sender may have an incentive to mislead the receiver. The solution concept we will use here is a *perfect Bayesian equilibrium* (PBE), instead of a *sequential equilibrium* (SE). The former is simpler than the latter. We do so without loss of generality

since the game of interest here is a finite Bayesian extensive game with observable actions. Every sequential equilibrium of the extensive game associated with a finite Bayesian extensive game with observable actions is equivalent to a perfect Bayesian equilibrium of the Bayesian extensive game, in the sense that they induce the same behavior and beliefs[4]. The formal definition of a PBE can be found in the Appendix, which is borrowed from Osborne and Rubinstein (p.233, 1994). In particular, we are interested in the following structure of the game **G** with two players where Player 1 has one of two types *strong* or *weak*:

$$\langle \Gamma, (\Xi_1 = \{\text{``strong''}, \text{``weak''}\}), (p_1(\text{``strong''}) = p), (u_i) \rangle \text{ with } \Gamma = \langle \{1, 2\}, H, P \rangle.$$

Figure 1 illustrates the structure of the game. We design this game such that it is as simple as possible but still contains all the essence of the signaling game. This is an econometric modelling of the *beer-quiche* game in Cho and Kreps (1987). In this game, we have two players. Player 1 has either of two types $\{strong, weak\}$ with the probability of being the strong type equal to p and knows her type. After observing her type, Player 1 moves first sending one of two messages $\{B, Q\}$ to Player 2. Then, Player 2 chooses an action "F" or "NF" after observing the signal sent by Player 1. After the play, a payoff is realized according to actions chosen by two players. The payoffs of this game have the following properties that are common in signaling games in general.

- The payoffs of Player 2 (*uninformed party*) given her action are determined by the type of Player 1 (*informed party*) not by Player 1's action (*signal*)
- Given Player 2's action, each type of Player 1 has bigger payoffs by choosing the signal corresponding to Player 1's true type. "B" is the signal for *strong* and "Q" is the signal for *weak* by construction ($\phi_{1s} > 0$, $\phi_{1w} > 0$)
- The *strong* type of Player 1 has an incentive to signal its true type
- The *weak* type of Player 1 has an incentive to mislead Player 2

The payoffs of the game are characterized by seven parameters:

- u_{1s}: difference of payoffs between "NF" and "F" outcomes for the *strong* type of Player 1
- u_{1w}: difference of payoffs between "NF" and "F" outcomes for the *weak* type of Player 1
- ϕ_{1s}: mimicking cost of the *strong* type of Player 1 (cost of signalling *falsely*)
- ϕ_{1w}: mimicking cost of the *weak* type of Player 1 (cost of signalling *falsely*)
- u_{2s}: difference of payoffs choosing between "NF" and "F" for Player 2 when Player 1 is *strong*
- u_{2w}: difference of payoffs choosing between "NF" and "F" for Player 2 when Player 1 is *weak*
- p: distribution of types

It is of our interest to identify these seven parameters that characterize the game from observed outcomes of actions. It is obvious that we can only identify up to three parameters nonparametrically if there is no exclusion or parametric restrictions since we have only three independent conditional probabilities out of four possible outcomes $\{B\&NF, B\&F, Q\&NF, Q\&F\}$. To ensure the identification[5], here we will adopt the following parametric specifications of the payoffs and we impose an exclusion restriction that X_2 contains at least one variable that does not enter in X_1, which has the support of at least three values.[6]

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[4]Note that the reverse statement is not true in general (see page 234-235, Osborne and Rubinstein (1994)). However, Fudenberg and Tirole (1991) note that for a finite Bayesian extensive game with two types or with two periods, every PBE is also equivalent to a sequential equilibrium. Thus, for the game of Figure 1&2, these two equilibrium concepts are equivalent each other.

[5]There are other possible ways of obtaining the identificaiton of seven parameters. We may even identify some parameters nonparametrically utilizing exclusion restrictions or functional form restrictions. This issue is one of our future research.

[6]For example, suppose we have no X_1 and suppose X_2 is a scalar random variable that can have one of three values $\{-1, 0, 1\}$. For each value of X_2, we have three linearly independent conditional probabilities. Thus, we have three sets of three linearly indepedent conditional probabilities (nine moment conditions), from which we can identify seven parameters of interest.

- $u_{1s} = \mu_s + X_1'\beta_s - \varepsilon_1$
- $u_{1w} = \mu_w + X_1'\beta_w - \varepsilon_1$
- $u_{2s} = X_2'\beta_2 - \varepsilon_2 + \phi_{2s}$
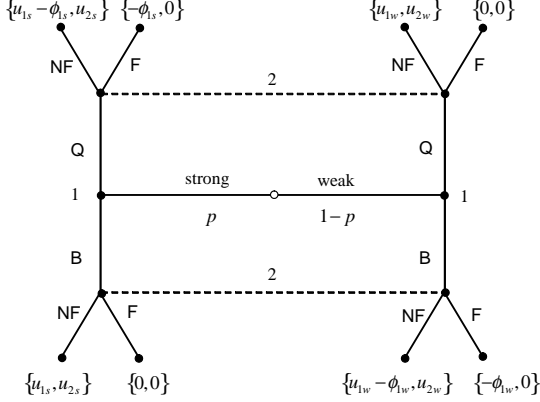- $u_{2w} = X_2'\beta_2 - \varepsilon_2 - \phi_{2w}$
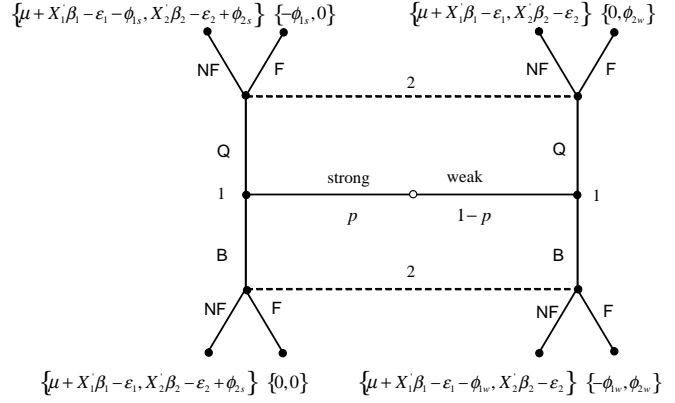

FIGURE 1. Structure of the Game

FIGURE 2. Parameterization of Payoffs

To make discussion simple as possible, here we only consider symmetric payoffs as in Figure 2[7] where we impose $\mu \equiv \mu_s = \mu_w$ and $\beta_1 \equiv \beta_s = \beta_w$.[8] The game with asymmetric payoffs are considered in Appendix B. Here we note that

- $\phi_{1s} > 0$ measures the potential cost of the *strong* type for mimicking the *weak* type given a fixed response of Player 2.
- $\phi_{1w} > 0$ measures the potential cost of the *weak* type for mimicking the *strong* type given a fixed response of Player 2.
- ϕ_{2s} and ϕ_{2w} measure degrees of Player 2's incentive to single out a particular type of Player 1.

Throughout the paper, we assume that ϕ_{2s} and ϕ_{2w} are not negative. Note that this assumption is innocuous in the sense that "strong" and "weak" types are just labels[9] unless we give some structure to it. By imposing $\phi_{2s} \geq 0$, we mean that Player 2 is more likely to be better off by singling out the *strong* type for the action "NF" and to be better off by singling out the *weak* type for the action "F".

The game tells that if Player 1 is the strong type and if Player 1 chooses "Q" and Player 2 chooses "NF", Player 1 obtains $\mu + X_1'\beta_1 - \varepsilon_1 - \phi_{1s}$ and Player 2 obtains $X_2'\beta_2 - \varepsilon_2 + \phi_{2s}$, respectively. If Player 1 is the strong type and if Player 1 chooses "Q" and Player 2 chooses "F", they earn $-\phi_{1s}$ and 0, respectively. Other playoffs can be read in the same way. It is noted that each player's payoffs are not only determined by her own action but also by the other player's action or type, which depicts interactions between players.

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[7] One might notice that the payoffs structure of Figure 2 is not nested by Figure 1 seeing the payoffs of Player 2 when she plays "F" and Player 1 is the weak type. However, note that what matters in the game is only the payoffs differential. We choose to use Figure 2 as an illustrational purpose to show that Player 2 is more likely to be better off by choosing "F" when Player 1 is the weak type.

[8] Here we assume that ϕ_{1s}, ϕ_{1w}, ϕ_{2s}, and ϕ_{2w} are constant but we may extend the model such that these paramters also depend on characteristics of Player 1 and Player 2, respectively.

[9] Note that depending on realizations of $\varepsilon_1 \& \varepsilon_2$ given $X_1 \& X_2$, the strong type can also have an incentive to mimic the weak type.

We let $X = (X_1', X_2')' \in \mathbb{R}^k$ with $k = k_1 + k_2$ and let $\varepsilon = (\varepsilon_1, \varepsilon_2)' \in \mathbb{R}^2$. We denote the vectors of parameters as $(\mu, \beta_1) \in \mathbb{R}^{k_1+1}$, $\beta_2 \in \mathbb{R}^{k_2}$, $\theta_{sg} = (\phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w})' \in (0, \infty)^2 \times [0, \infty)^2$, and let $(\theta, p) = (\mu, \beta_1', \beta_2', \theta_{sg}', p)' \in \mathbb{R}^{k+1} \times (0, \infty)^2 \times [0, \infty)^2 \times (0, 1)$. These parameters of interest are perfectly known to players but unknown to econometricians.

Note that this stochastic payoffs game is different from the deterministic payoffs game in that by adding the unobserved heterogeneity $(\varepsilon_1, \varepsilon_2)$, we allow such cases that players with the same observed characteristics (X_1, X_2) can show different outcomes of the game. We also note that the payoffs structure of the game $\mathbf{G}$ is somewhat restrictive in the sense that we let the following two payoffs[10] be the same. One is the payoff of the *strong* type Player 1 when she chooses "B" and Player 2 plays "NF". The other is the payoff of the *weak* type Player 1 when she chooses "Q" and Player 2 chooses "NF". This structure may be justified in some cases but might be too restrictive in general. We relax this restriction in Appendix B. It turns out that the games with/without asymmetric payoffs are very similar in terms of equilibrium characterization.

2.1 The Information Structure (IS)

The game we study has *incomplete information* since players do not have exact knowledge about the payoffs of their opponents. It is also a signaling game since the true type of Player 1 is only known to Payer 1 herself and Player 1 signals her type to Player 2 by choosing some action.

Assumption 2.1 *(IS)*

1 Player 1 knows her true type but Player 2 knows only the distribution of Player 1's types (p is known to Player 2).

2 The realizations of (X_1, ε_1) and (X_2, ε_2) are perfectly observed by both Player 1 and Player 2.

3 ε_1 and ε_2 are pure shocks commonly observed by Player 1 and Player 2. They are independent of each other and of any other variables in the game. ε_1 is also independent of the type of Player 1.

4 Players' actions and beliefs constitute a Perfect Bayesian Equilibrium (Sequential Equilibrium). Whenever there exist multiple equilibria, only one equilibrium is chosen out of these according to some equilibrium refinements. Players are assumed to play actions and hold beliefs about this unique equilibrium.

Note that the generic uniqueness of the PBE is ensured by some refinements of the equilibrium concept.

2.2 Stochastic Assumptions (SA-1)

We impose the following distributional assumptions on the random variables of the game. We first consider the simplest structure and generalize it later. Hereafter we denote the support of a random variable by $\mathbb{S}(\cdot)$.

Assumption 2.2 *(SA-1)*

1 ε_1 and ε_2 are continuously distributed, statistically independent of each other and of X.

2 The cdf's of ε_1 and ε_2 are continuous and denoted by $G_1(\varepsilon_1)$ and $G_2(\varepsilon_2)$ with corresponding density functions $g_1(\varepsilon_1)$ and $g_2(\varepsilon_2)$, respectively. The density functions are assumed to be bounded and strictly positive on their supports $\mathbb{R}$. The density functions do not depend on the model parameter (θ, p) nor on the type of Player 1.

3 Both X_1 and X_2 can be continuous or discrete random variables. Both X_1 and X_2 are independent of the type of Player 1. We denote the density of X_1 and X_2 as $f_{X_1}(\cdot)$ and $f_{X_2}(\cdot)$, respectively. Neither $f_{X_1}(\cdot)$ or $f_{X_2}(\cdot)$ depends on the structural parameter (θ, p).

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[10]Actually, the payoffs differences beween the outcomes of "NF" and "F" due to the normalization.

4 X_2 contains at least one variable that does not enter in X_1, which has the support of at least three values.

By imposing Assumption **SA-1.1**, we ensure that Player 2's equilibrium beliefs are constructed conditional on variables observed by the econometrician. The exclusion restriction of Assumption **SA-1.4** is easily satisfied for the game model we consider since there exist inherent exclusions between X_1 and X_2. In other words, X_1 and X_2 are different variables for a typical signaling game. We will strengthen these stochastic assumptions to ensure the validity of the econometric modelling in the later section.

3   Refinement and Uniqueness of Equilibrium

In this section we characterize equilibria of the game under PBE and then show we can achieve the uniqueness of equilibrium using a refinement of Cho and Kreps (1987). For the game we study, it turns out that we have multiple equilibria for some realizations of payoffs, which can be removed adopting a stronger equilibrium concept as an equilibrium selection. The disadvantage of this approach noted in the literature is that there is neither generally accepted or empirically testable procedure to determine which equilibrium will be played among multiple equilibria, especially for simultaneous move games. However, we note that an equilibrium selection of a signaling game is more acceptable than that of a simultaneous game, at least theoretically, which may be a justification of our approach but we relax this in Section 6.

We introduce some notation here. Let $u_i(t_1; A_1, A_2)$ denote the payoffs of player $i \in \{1, 2\}$ when the true type of Player 1 is $t_1 \in \{t_s \equiv strong, t_w \equiv weak\}$, Player 1 chooses action (signal) $A_1 \in \{B, Q\}$ and Player 2 chooses action $A_2 \in \{NF, F\}$. We also let A_{1t_1} denote the action taken by a particular type of Player 1. With (A_1, A_1'), we mean $(A_{1t_s}, A_{1t_w}) = (A_1, A_1')$ where $A_1, A_1' \in \{B, Q\}$. Note $p = \Pr(t_1 = t_s)$ is the prior belief of Player 2 on the type of Player 1. This is also the population distribution of types. We let $E_1[u_1(t_1; A_1, A_2)]$ be the expected payoff of Player 1 with type t_1 based on Player 1's information. $\mu_2(t_1|A_1)$ denotes the posterior belief of Player 2 on the type of Player 1 after Player 2 observes the action (signal) of A_1. We also let $Y_{2|A_1}$ be an indicator function that has the value 1 when Player 2 chooses the action "NF" after observing the signal A_1. Similarly, $A_{2|A_1}$ denotes the action of Player 2 after observing A_1. Throughout this paper, we will use this notation. In the Appendix D.2, we determine regions of $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$, where each PBE exists given a realization of X. Each PBE is obtained using the four conditions of Definition D.1 in the Appendix: *sequential rationality, correct initial belief, action-determined beliefs*, and *Bayesian updating*.

Figure 3[11] illustrates regions of $(\varepsilon_1, \varepsilon_2)$ where a particular equilibrium is supported[12] in terms of observations, assuming $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). It is noted that under $\varepsilon_2 > X_2'\beta_2 + \phi_{2s}$, Player 2 is better off by choosing F regardless of Player 1's type or action and will choose F. Thus, Player 1 will choose a signal corresponding to her type since given Player 2's action, Player 1 is better off by choosing the signal that corresponds to her type. Similarly, under $\varepsilon_2 < X_2'\beta_2 - \phi_{2w}$, Player 2 is better off by choosing NF no matter what and thus under this region, Player 1 is willing to reveal her true type. We also note that in many empirical studies, researchers tend to focus on the separating equilibrium where Player 1 reveals her type and Player 2 chooses different actions according to different types of Player 1

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[11]For the game of Figure I in Cho and Kreps (1987), we have two pooling PBE. In one equilibrium, both types of Player 1 choose B, and Player 2 does not fight if she observes B and she fights if she oberves Q with out-of-equilibrium belief $\mu_2(t_s|Q) \leq 0.5$. In the other equilibrium, both types of Player 1 choose Q, Player 2 chooses not to fight if she observes Q and she fights if she observes B with out-of-equilibrium belief $\mu_2(t_s|B) \leq 0.5$. We note that the example of Cho and Kreps (1987) corresponds to region $\boldsymbol{S}_5 \equiv \{(\varepsilon_1, \varepsilon_2)|\varepsilon_1 < \mu + X_1'\beta_1 - \phi_{1w}$ and $X_2'\beta_2 - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}\}$ in Figure 3 and 4.

[12]To define a PBE, we also specify the out-of-equilibrium belief that supports a particular equilibrium. A specific out-of-equilibrium belief for each PBE of the game can be found in the Mathematical Appendix. Here we do not present such out-of-equilibrium belief to make discussion simple.

(such as region $\boldsymbol{S}_3 \equiv \{(\varepsilon_1, \varepsilon_2) | \mu + X_1'\beta_1 - \phi_{1w} < \varepsilon_1 < \mu + X_1'\beta_1 + \phi_{1s}$ and $X_2'\beta_2 - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$[13] in Figure 3) by imposing some conditions or by simply asserting a separating equilibrium is more reasonable. However, Figure 3 illustrates that other kinds of equilibria can arise depending on the realizations of $(\varepsilon_1, \varepsilon_2)$. For example, in region $\{\varepsilon_1 < \mu + X_1'\beta_1 - \phi_{1w}$ and $X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$, we have a semi-separating equilibrium where the strong type plays B and the weak type mixes between B and Q. Similarly, in region $\{\varepsilon_1 > \mu + X_1'\beta_1 + \phi_{1s}$ and $X_2'\beta_2 - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}\}$, we have another semi-separating equilibrium where the weak type plays Q and the strong type mixes between B and Q.

The following two theorems are about the existence of PBE for all regions of $(\varepsilon_1, \varepsilon_2)$ given X and conditions to achieve uniqueness of equilibrium.

Theorem 3.1 *(Existence of Equilibrium)*
*Suppose Assumptions **IS** and **SA-1** hold. Suppose also that $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). Then, there exist PBE for all regions of $(\varepsilon_1, \varepsilon_2)$ given $X = x$.*

See the Appendix D.2 for the proof. We note that there are regions where multiple equilibria arise. However, using the refinement of Cho and Kreps (1987), we can achieve uniqueness of equilibria for each region of $(\varepsilon_1, \varepsilon_2)$ given X. Figure 3[14] shows that there exist three regions where we have multiple equilibria. In region $\boldsymbol{A}_a \equiv \{(\varepsilon_1, \varepsilon_2) | \varepsilon_1 > \mu + X_1'\beta_1 + \phi_{1w}$ and $X_2'\beta_2 + (\phi_{2s} + \phi_{2w})p - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$, we have two equilibria. One is pooling (B, B) with $A_{2|B} = F$ and the other is pooling (Q, Q) with $A_{2|Q} = F$ or separating equilibrium (when $\phi_{1w} < \phi_{1s}$). In region $\boldsymbol{A}_b \equiv \{\varepsilon_1 < \mu + X_1'\beta_1 - \phi_{1s}$ and $X_2'\beta_2 - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}\}$, we also have two equilibria. One is pooling (Q, Q) with $A_{2|Q} = NF$ and the other is pooling (B, B) with $A_{2|B} = NF$ or separating equilibrium. In Appendix A, using the intuitive criterion of Cho and Kreps (1987)[15], we show that the pooling (B, B) fails the refinement in region $\boldsymbol{A}_a$, the pooling (Q, Q) fails the refinement in $\boldsymbol{A}_b$[16]. Figure 4 illustrates the uniqueness of equilibrium based on this refinement result.

We need to distinguish an equilibrium from its realized outcomes. For example, in region $\boldsymbol{S}_1 \equiv \{(\varepsilon_1, \varepsilon_2) | \varepsilon_1 \in \mathbb{R}$ and $\varepsilon_2 > X_2'\beta_2 + \phi_{2s}\}$, we have one equilibrium where Player 1 plays the separating equilibrium with (B, Q) and Player 2 chooses F. However, in terms of realized outcomes, we have two possible outcomes in this region. With probability p (when Player 1 is the strong type), we will observe the $B\&F$ combination but we can also observe the the $Q\&F$ combination with probability $1 - p$ (when Player 1 is the weak type). Likewise even though we have only one equilibrium (semi-separating where the strong type plays B and the weak type mixes between B and Q) in region $\boldsymbol{S}_2 \equiv \{(\varepsilon_1, \varepsilon_2) | \varepsilon_1 < \mu + X_1'\beta_1 - \phi_{1w}$ and $X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$, we can observe all four possible outcomes with certain probabilities, respectively.

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[13] We note that this region gets larger as ϕ_{1s}, ϕ_{1w}, ϕ_{2s}, and ϕ_{2w} become larger. This implies that the larger the cost of mimicking and the larger Player 2's incentives of singling out a particular type of Player 1, the larger the region that supports this separating equilibrium.

[14] Figure 3 depicts the case $\phi_{1w} > \phi_{1s}$ but this is not necessary at all.

[15] The intuitive criterion is based on the concept of "equilibrium dominance" which tells that a certain type should not be expected to use a certain strategy. For example, in region $\boldsymbol{S}_4 \equiv \{(\varepsilon_1, \varepsilon_2) | \varepsilon_1 > \mu + X_1'\beta_1 + \phi_{1s}$ and $X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) + \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$, Player 1 plays a pooling equilibrium strategy with (B, B) or (Q, Q). However, for the pooling equilibrium with (B, B) under $\boldsymbol{S}_4$, it is not reasonable to believe that a deviation is played by the *strong* type because she is better off by choosing B no matter what Player 2 chooses (see Appendix). Thus, Player 2 will think a deviation is played by the *weak* type for sure when she observes Q. Under this refined belief, now we can eliminate the pooling with (B, B) in region $\boldsymbol{S}_4$.

[16] For the game of Figure I in Cho and Kreps (1987), the pooling PBE with Q fails the intuitive criterion This corresponds to region $\boldsymbol{S}_5$ in Figure 3 and 4.
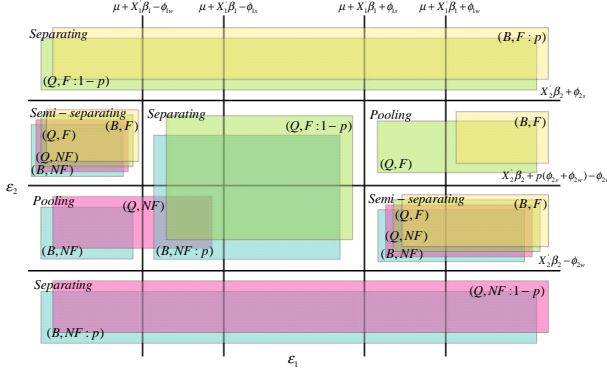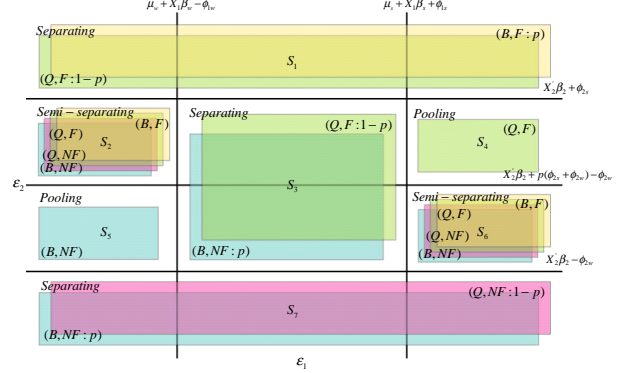
FIGURE 3. Multiple Equilibria of the Game

FIGURE 4. Uniqueness of Equilibrium

Theorem 3.2 *(Uniqueness of Equilibrium)*
*Suppose Assumptions **IS** and **SA-1** hold and that $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). Suppose each Player plays only one equilibrium that survives the refinement of Cho and Kreps (1987), when there exist multiple equilibria. Then, there exist unique equilibrium for each region of $(\varepsilon_1, \varepsilon_2)$ given X.*

Theorem 3.2 enables us to obtain a well-defined likelihood function. One may think that we cannot identify μ separately from ϕ_{1s} and ϕ_{1w} under unique equilibrium as Figure 4 suggests[17]. However, we actually can identify μ separately from ϕ_{1s} and ϕ_{1w} since ϕ_{1s} and ϕ_{1w} appear separately from μ in the model conditional probabilities due to the semi-separating equilibria (see Appendix C). From the result of Theorem 3.2, we can define the conditional probabilities for four possible observed outcomes.

$$P_{jl}(X, \theta, p) = \Pr(Y_1 = j, Y_2 = l | X) \text{ for } j, l \in \{0, 1\}.$$

The specific forms of those conditional probabilities are provided in Appendix C. Now using these conditional probabilities, one can estimate the parameter θ, using the conditional ML method such that

$$\left(\widehat{\theta}_{CML}, \widehat{p}_{CML}\right) = \underset{\theta \in \Theta, p \in (0,1)}{\operatorname{argmax}} \frac{1}{n} \sum_{i=1}^{n} \log L\left(y_i | x_i, \theta, p\right) \tag{1}$$

where

$$
\begin{aligned}
\log L\left(y_i | x_i, \theta, p\right) =\ & y_{1i} y_{2i} \log P_{11}(x_i, \theta, p) + y_{1i}\left(1 - y_{2i}\right) \log P_{10}(x_i, \theta, p) \\
& + \left(1 - y_{1i}\right) y_{2i} \log P_{01}(x_i, \theta, p) + \left(1 - y_{1i}\right)\left(1 - y_{2i}\right) \log P_{00}(x_i, \theta, p).
\end{aligned}
$$

The estimator $\left(\widehat{\theta}_{CML}, \widehat{p}_{CML}\right)$ will be consistent and asymptotically normal under suitable conditions. Those conditions can be found in Newey and McFadden (1994), for example. Note that this conditional ML estimator is identical to the ML estimator since the density function of X does not depend on the parameter (θ, p) by Assumption **SA-1.3**.

---

[17] We note that multiplicity of equilibria may help the identification of μ since there are more critical values (four vertical lines in Figure 3 compared to two vertical lines in Figure 4) of ε_1 that switch the kind of equilibrium. We also note that addtional incomplete information such that Player 1 cannot observe ε_2 (see the Mathematical Appendix E) can also help the identification since there are four critical values of ε_1 that switch the kind of equilibrium and these critical lines vary according to different values of X_2. This is an interesting result.

4 Introducing Public Information about the Type

We consider public signals about the type of Player 1. We denote such signals by Z, which are observable to all the players of the game and econometricians. Until now, we have assumed that X_1 is independent of the type of Player 1. However, it is likely that at least some of observed characteristics of Player 1 will reveal information regarding the type of Player 1. In the *beer or quiche* game story, characteristics of Player 1 such as muscle intensity, height, or age will tell how likely Player 1 is the *strong* type. Thus, Z can include all the variables in X_1 or a subset of X_1. In the job signaling game (Spence (1974)), parents' education is a possible public signal for the ability of a job candidate.

For a public signal, we require that Player 1 cannot strategically choose the signal Z when a game is played[18] or at least Player 1 does not have an incentive to do so. This means that in the game $\boldsymbol{G}$, only the action A_1 plays the role of the strategic signal. The public signal $Z \in \mathbb{R}^{\dim(Z)}$ has a mixing distribution $F_Z(\cdot)$ with a mixing variable p. We let the density of Z as

$$f_Z(z) = pf_{(st)}(z) + (1-p)f_{(we)}(z) \tag{2}$$

where p is known to the players of the game but not known to econometricians. Player 2 has an incentive to use these signals while playing the game. When players play a separating equilibrium, these additional signals have no additional information for Player 1's type since Player 1's type is perfectly inferred from her action. When the players play a pooling equilibrium, Player 2 will use these additional signals to update her belief on Player 1's type using the Bayes' rule. Therefore, under a separating equilibrium, we have

$$\mu_2(t_1 = t_s | A_1, Z) = \mu_2(t_1 = t_s | A_1)$$

but under a pooling equilibrium, we have

$$\mu_2(t_1 = t_s | A_1, Z = z) = \frac{pf_{(st)}(z)}{pf_{(st)}(z) + (1-p)f_{(we)}(z)} \tag{3}$$

which is also the conditional probability of being *strong* type given $Z = z$. We let $p(z)$ denote this conditional probability, $Pr(t_1 = t_s | Z = z) = \frac{pf_{(st)}(z)}{pf_{(st)}(z) + (1-p)f_{(we)}(z)}$. Thus, we have $p(Z) = \mu_2(t_1 = t_s | A_1, Z)$ under a pooling equilibrium. Note that equation (3) and $p(z)$ become the prior p, when $f_{(st)}(z) = f_{(we)}(z)$ (no mixture). We will maintain the following assumptions:

Assumption 4.1 (IS-A)

    1 Assumption **IS** holds.
    2 The public signal Z about the types of Player 1 is perfectly known to both Player 1 and Player 2.

Assumption 4.2 (SA-1A)

    1 ε_1 and ε_2 are continuously distributed, statistically independent of each other and of X and Z.
    2 The cdf's of ε_1 and ε_2 are continuous and denoted by $G_1(\varepsilon_1)$ and $G_2(\varepsilon_2)$ with corresponding density functions $g_1(\varepsilon_1)$ and $g_2(\varepsilon_2)$, respectively. The density functions are assumed to be bounded and strictly positive on their supports $\mathbb{R}$. The density functions do not depend on the model parameter (θ, p) or on the type of Player 1.
    3 Both X_1 and X_2 can be continuous or discrete random variables. Both $X_1 - Z$ and X_2 are independent of the type of Player 1. We denote the density of $X_1 - Z$ and X_2 as $f_{X_1-Z}(\cdot)$ and $f_{X_2}(\cdot)$, respectively. Neither $f_{X_1-Z}(\cdot)$ or $f_{X_2}(\cdot)$ depends on the structural parameter (θ, p).

---

[18] It means that in the *beer-quiche* game, Player 1 cannot work out her muscle when the game is played.

4 Z is a continuously distributed random vector with the mixing density $f_Z(z) = pf_{(st)}(z) + (1 - p)f_{(we)}(z)$. Neither $f_{(st)}(z)$ or $f_{(we)}(z)$ depends on the structural parameter (θ, p).

5 X_2 contains at least one variable that does not enter in X_1, which has the support of at least three values.

In the game **G** under Assumptions **IS-A** and **SA-1A**, it is not difficult to see that we will obtain exactly the same equilibria under Assumptions **IS** and **SA-1** except for replacing $\mu_2(t_1 = t_s|A_1)$ with $\mu_2(t_1 = t_s|A_1, Z = z)$. This means that whenever Player 2's belief is involved, we replace p with $p(z)$. Note that we should distinguish the population distribution of types ($\Pr(t_1 = t_s) = p$) from the posterior belief of Player 2 after observing the signal $Z = z$ under a pooling equilibrium, which should also be the conditional probability of being the *strong* type given $Z = z$ in equilibrium. Regardless of the public signal Z, the overall distribution of types (p) is fixed by nature. We apply the refinement of Cho and Kreps (1987) to these games with public signal Z and obtain the same uniqueness of equilibrium with Theorem 3.2.[19]

We will use a logistic specification for the posterior belief of Player 2 under a pooling equilibrium noting that equation (3) can be rewritten as

$$p(z) = \frac{pf_{(st)}(z)}{pf_{(st)}(z) + (1 - p)f_{(we)}(z)} = \frac{\frac{p}{1-p}\frac{f_{(st)}(z)}{f_{(we)}(z)}}{1 + \frac{p}{1-p}\frac{f_{(st)}(z)}{f_{(we)}(z)}} \tag{4}$$

and thus, the belief only depends on p and the ratio $f_{(st)}(z)/f_{(we)}(z)$. The relationship in (4) means that for updating, we do not need to know $f_{(st)}$ and $f_{(we)}$ individually but only the ratio between these two is necessary.

Therefore, by letting $h^o(z) = \log\left(\frac{f_{(st)}(z)}{f_{(we)}(z)}\right)$, we can rewrite (3) as a logistic specification with $L(\cdot) = \frac{\exp(\cdot)}{1+\exp(\cdot)}$,

$$p(Z = z) = \frac{\exp\left(\log(\frac{p}{1-p}) + h^o(z)\right)}{1 + \exp\left(\log(\frac{p}{1-p}) + h^o(z)\right)} = L(\log(\frac{p}{1-p}) + h^o(z)) \tag{5}$$

recalling that the posterior belief of Player 2 under a pooling equilibrium equals to the conditional probability of being *strong* type given $Z = z$.

This specification reflects the separability of the mixing variable and the component functions. If there is no mixture, we will have $h^o(z) = \log\left(f_{(st)}(z)/f_{(we)}(z)\right) = 0$. It suggests that by examining whether or not $h^o(z) = 0$, we may test the existence of a mixing distribution.

5 Estimation of the Signaling Game

In this section, we provide several estimation methods for the game **G** with public signals. We note that the mixing distribution of public signals is nonparametrically specified. The information structure of **IS-A** is maintained but the stochastic assumptions are strengthened to facilitate the estimation. To preserve uniqueness of equilibrium, we also maintain the assumption that each player plays only one equilibrium that survives the refinement of Cho and Kreps (1987), when there exist multiple equilibria. We also let d_l denote the dimension of a vector l. We let Y_1 denote the indicator function that has the value one when Player 1 chooses "B" such that $Y_1 \equiv \mathbf{1}(A_1 = B)$. Similarly, we let $Y_2 \equiv \mathbf{1}(A_2 = NF)$. We also let $Y = (Y_1, Y_2)$. We let $C(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$, and so on denote generic positive constants or functions. For a

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[19]For the game with additional incomplete information presented in the Mathematical Appendix E, we also obain the same result with Theorem E.3.

positive number k, we let $\underline{k}$ denote the largest integer smaller than k. Finally, we let the upper case stand for a random variable and the lower case stand for a realization of it. We use the subscript "$_0$" to denote the true value of parameters. Throughout the paper, we assume that econometricians observe the realizations of the random variables $X_1 \cup Z$, X_2, Y_1, and Y_2 but do not observe ε_1 or ε_2. We let $W \equiv X \cup Z$ and let $\theta = (\mu, \beta_1', \beta_2', \phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w})' \in \mathbb{R}^{1+k_1+k_2} \times (0, \infty)^2 \times [0, \infty)^2$, $\alpha^o = (\theta, p, h^o) \in \mathcal{A}^o \equiv \Theta \times (0, 1) \times \mathcal{H}^o$. The α^o is the parameter of interest that econometricians want to estimate.

5.1 Regularity Conditions

Now we impose stronger conditions than **SA-1A** by adding some "smoothness" conditions for $G_1(\cdot)$ and $G_2(\cdot)$.

Assumption 5.1 (SA-2)

1. ε_1 and ε_2 are continuously distributed, statistically independent of each other and of $W \equiv X \cup Z$.

2. The cdf's of ε_1 and ε_2 are continuous and denoted by $G_1(\varepsilon_1)$ and $G_2(\varepsilon_2)$ with corresponding density functions $g_1(\varepsilon_1)$ and $g_2(\varepsilon_2)$, respectively. The density functions are assumed to be bounded and strictly positive on their supports $\mathbb{R}$ ($\mathbb{S}(\varepsilon_1) = \mathbb{S}(\varepsilon_2) = \mathbb{R}$). The density functions do not depend on the model parameter α^o or on the type of Player 1.

3. $G_1(\varepsilon_1)$ and $G_2(\varepsilon_2)$ are $\underline{\nu}+3$ times continuously differentiable with bounded $\underline{\nu}+3$ derivatives everywhere in $\mathbb{S}(\varepsilon_1) = \mathbb{S}(\varepsilon_2) = \mathbb{R}$. The density of g_1 and g_2 are known to Player 1 but an econometrician knows g_1 and g_2 up to a finite dimensional parameter[20], respectively.

4. Both X_1 and X_2 can be continuous or discrete random variables. Both $X_1 - Z$ and X_2 are independent of the type of Player 1. We denote the density of $X_1 - Z$ and X_2 as $f_{X_1-Z}(\cdot)$ and $f_{X_2}(\cdot)$, respectively. Neither $f_{X_1-Z}(\cdot)$ or $f_{X_2}(\cdot)$ depends on the model parameter α^o.

5. In particular, we assume that $\mathbb{S}(X_1) \subset \mathbb{R}^{k_1}$ and $\mathbb{S}(X_2) \subset \mathbb{R}^{k_2}$ are compact.

6. Z is a continuously distributed random vector with density $f_Z(z)$. $\mathbb{S}(Z)$ is compact with nonempty interior. The density $f_Z(z) = pf_{(st)}(z) + (1-p)f_{(we)}(z)$ is bounded and bounded away from zero. Neither $f_{(st)}(z)$ or $f_{(we)}(z)$ depends on the structural parameter (θ, p).

7. X_2 contains at least one variable that does not enter in X_1, which has the support of at least three values.

All the assumptions are standard in the literature. In particular, Assumption **SA-2.7** is an order condition for the identification of model parameters. Again we note that the exclusion restriction of Assumption **SA-2.7** is easily satisfied due to the inherent exclusions between X_1 and X_2. X_1 and X_2 are allowed to include continuous and/or discrete random variables. Noting that some or all of variables in X_1 can be included in Z. We let $W = X \cup Z$ and denote the joint pdf as $f_W(w)$. The joint density $f_W(w)$ is unknown to the econometrician except for some smoothness conditions.

5.2 Identification and Estimation

Now we let $\theta = (\mu, \beta_1', \beta_2', \phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w})' \in \mathbb{R}^{1+k_1+k_2} \times (0, \infty)^2 \times [0, \infty)^2$, $\alpha = (\theta, h) \in \mathcal{A} \equiv \Theta \times \mathcal{H}$ where $h = \log\left(\frac{p}{1-p}\right) + h^o$. Then, we let $\mathcal{L}_{Y_1Y_2}(W, \alpha)$ denote the conditional probabilities of observed outcomes such that

$$\mathcal{L}_{y_1y_2}(W, \alpha) \equiv \Pr(Y_1 = y_1, Y_2 = y_2 | W, \alpha) \text{ for } (y_1, y_2) = \{(1,1), (1,0), (0,1), (0,0)\}$$

---

[20] Here we assume that those parameters are known, i.e. the functional forms of $g_1(\cdot)$ and $g_2(\cdot)$ are known to simplify our notation without loss of generality as Aradillas-Lopez (2005).

where $\Pr(Y_1 = y_1, Y_2 = y_2|W, \alpha)$ takes the rather complicated form and is defined in Appendix C. Recall that we need to distinguish the population probability p from the posterior belief $p(Z)$ of Player 2, which is also the conditional probability of being *strong* type conditional on Z. We note that one may think we cannot identify p without a restriction on h^o since the conditional probabilities depend on $p(Z)$, not on p separately. However, it turns out that we can still identify p using the relationship

$$E[p(Z)] = \int p(z)f_Z(z)dz = \int \frac{pf_{(st)}(z)}{f_Z(z)}f_Z(z)dz = p\int f_{(st)}(z)\,dz = p. \tag{6}$$

We will discuss this issue later (see Section 5.3). Now we let $\alpha_0 \equiv (\theta_0, h_0)$ denote the true value of α. Later we show that the following condition is sufficient for identification of α_0.

Assumption 5.2 (SA-3) *Conditional on W, if $\alpha \neq \alpha_0$ for $\alpha, \alpha_0 \in \mathcal{A}$, then*

$$\Pr\left(\mathcal{L}_{y_1 y_2}(W, \alpha) \neq \mathcal{L}_{y_1 y_2}(W, \alpha_0)\right) > 0$$

for $(y_1, y_2) = \{(1,1), (1,0), (0,1), (0,0)\}$.

Now we let

$$
\begin{aligned}
&l(Y|W, \alpha) \equiv l(Y|W, \theta, h) \\
&= \left[\begin{array}{c} Y_1 Y_2 \log \mathcal{L}_{11}(W, \theta, h) + Y_1 (1 - Y_2) \log \mathcal{L}_{10}(W, \theta, h) \\ + (1 - Y_1) Y_2 \log \mathcal{L}_{01}(W, \theta, h) + (1 - Y_1)(1 - Y_2) \log \mathcal{L}_{00}(W, \theta, h) \end{array} \right]
\end{aligned}
$$

denote the single observation conditional log likelihood function. We will estimate the parameters of interest using the conditional sieve ML based on this conditional likelihood function. In this estimation, we approximate the unknown function h with some sieves. For this purpose, we need to restrict the space of functions $\mathcal{H}$ where h_0 belong. We consider the Hölder space $\Lambda^{\nu_1}(\mathbb{S}(Z))$ with order $\nu_1 > 0$ as Ai and Chen (2003). The Hölder space is a space of functions $g : \mathbb{S}(Z) \longrightarrow \mathbb{R}$ such that the first ν_1 derivatives are bounded, and the $\underline{\nu_1}$-th derivatives are Hölder continuous with the exponent $\nu_1 - \underline{\nu_1} \in (0, 1]$, where $\underline{\nu_1}$ is the largest integer smaller than ν_1. The Hölder space becomes a Banach space when endowed with the Hölder norm:

$$||g||_{\Lambda^{\nu_1}} = \sup_z |g(z)| + \max_{a_1 + a_2 + \ldots a_{d_z} = \underline{\nu_1}} \sup_{z \neq z'} \frac{|\nabla^a g(z) - \nabla^a g(z')|}{(||z - z'||_E)^{\nu_1 - \underline{\nu_1}}} < \infty,$$

where $\nabla^a g(z) \equiv \frac{\partial^{a_1 + a_2 + \ldots a_{d_z}}}{\partial z_1^{a_1} \ldots \partial z_{d_z}^{a_{d_z}}} g(z)$. The Hölder ball (with radius C_1) $\Lambda_{C_1}^{\nu_1}(\mathbb{S}(Z))$ is defined accordingly as

$$\Lambda_{C_1}^{\nu_1}(\mathbb{S}(Z)) \equiv \{g \in \Lambda^{\nu_1}(\mathbb{S}(Z)) : ||g||_{\Lambda^{\nu_1}} \leq C_1 < \infty\}.$$

In the literature, it is known that functions in $\Lambda_{C_1}^{\nu_1}(\mathbb{S}(Z))$ can be approximated well by various sieves such as power series, Fourier series, splines, and wavelets. We let $\mathcal{H} = \Lambda_{C_1}^{\nu_1}(\mathbb{S}(Z))$. In particular, we approximate $h(\cdot)$ by power series. According to Theorem 8, p.90 in Lorentz (1986) (Also see Timan (1963)), if a function f is s-times continuously differentiable, then there exist a K-vector γ_K and a triangular array of polynomials $R^K(z)$ (Particularly $E[R^K(Z)R^K(Z)'] = I_K$) on the compact set $\mathcal{Z}$ such that

$$\sup_{z \in \mathcal{Z}} |f(z) - R^K(z)'\gamma_K| \leq C_1 K^{-\frac{s}{d_z}}. \tag{7}$$

We consider the tensor-product power series[21], $\mathcal{H}_n$, as a sieve space such that

$$\mathcal{H}_n = \{h(z)|h(z) = R^K(z)'\pi \text{ for all } \pi \text{ satisfying } ||h||_{\Lambda^{\nu_1}} \leq C_1\} \tag{8}$$

---

[21] For comprehensive discussion of tensor-product linear sieves including power series, see Chen (2005).

where we let the first element of $R^K(z)$ equal to the constant 1.[22]

Then, from the result of (7), we have a $h_K(\cdot) \equiv R^K(\cdot)'\pi_K$ such that

$$\sup_{z \in \mathbb{S}(Z)} |h_0(z) - R^K(z)'\pi_K| = O\left(K^{-\frac{\nu_1}{d_z}}\right). \tag{9}$$

We obtain the estimator $\widetilde{\theta}_n$ and $\widetilde{h}_n$ as

$$\widetilde{\alpha}_n \equiv \left(\widetilde{\theta}_n, \widetilde{h}_n\right) = \operatorname*{argmax}_{(\theta,h) \in \mathcal{A}_n \equiv \Theta \times \mathcal{H}_n} \widehat{Q}(\theta, h) \equiv \frac{1}{n}\sum_{i=1}^n l(y_i|w_i, \theta, h(z_i)). \tag{10}$$

We call $\widetilde{\alpha}_n$ the exact sieve conditional ML estimator. Because the complexity (in the sense defined in the Appendix F.3) of the sieve space $\mathcal{A}_n$ increases with n and because the maximizer of (10) is often obtained numerically, we do not require the maximization of $\widehat{Q}(\theta, h)$ over $\mathcal{A}_n$ need to be exact. An approximated estimator, $\widehat{\alpha}_n$ is enough for the asymptotic results, such that

$$\widehat{Q}(\widehat{\alpha}_n) \geq \sup_{\alpha \in \mathcal{A}_n} \widehat{Q}(\alpha) - O_p(\varepsilon_n), \text{ with } \varepsilon_n = o(1). \tag{11}$$

We call $\widehat{\alpha}_n$ in (11) approximate sieve ML estimator. Note that we choose the order of ε_n such that it can justify a desirable asymptotics result. Now define

$$Q(\alpha) \equiv Q(\theta, h) = E[l(Y|W, \theta, h)]. \tag{12}$$

The following lemma shows that if Assumption **SA-3** holds, then $l(Y|W, \alpha)$ satisfies an information inequality result which is useful to prove the consistency of our proposed estimator $\widehat{\alpha}_n$ in (11) together with the uniform convergence of $\widehat{Q}(\alpha)$ to $Q(\alpha)$.

Lemma 5.1 *(Identification)*
*Suppose Assumptions **IS-A** and **SA-2** hold. Further suppose Assumption **SA-3** holds. Then, $Q(\alpha) < Q(\alpha_0)$ for all $\alpha \neq \alpha_0 \in \mathcal{A}$.*

The proof can be found in the Appendix.

It is worthwhile to note that the conditional ML estimator $\widetilde{\alpha}_n$ does not use all the information we have. In particular, we do not utilize the information about the mixing distribution of Z. The conditional ML will be numerically identical to the ML if the density of W does not depend on the parameter α. However, we know that the density of Z depends on h^o. If we assume that Z is independent with $(X_1 - Z) \cup X_2$ (or we already eliminate such dependence), the full ML estimator will be given by

$$\widetilde{\alpha}_{Full}^o = \operatorname*{argmax}_{(\theta,p,h^o) \in \mathcal{A}_n \equiv \Theta \times (0,1) \times \mathcal{H}_n} \left(\frac{1}{n}\sum_{i=1}^n \log(p\exp(h^o(z_i)) + (1-p)) + \frac{1}{n}\sum_{i=1}^n l(y_i|w_i, \theta, p, h^o(z_i))\right) \tag{13}$$

noting that neither $f_{X_1 - Z}$ or f_{X_2} depends on α^o, that

$$\log\left(pf_{(st)}(z) + (1-p)f_{(we)}(z)\right) = \log f_{(we)}(z) + \log(p\exp(h^o(z_i)) + (1-p)),$$

and that $f_{(we)}(z)$ does not depend on α^o. Therefore, the conditional ML estimator $\widetilde{\alpha}_n$ is different from the corresponding ML and will be less efficient (see Gourieroux and Monfort (1995), Section 7.5.3) since it is obtained by dropping the first term in the log likelihood function. In the Appendix G, we discuss how to recover this lost information using a *pseudo EM algorithm*. Note that in the ML, we may estimate all of the

---

[22] Noting $h = \log\left(\frac{p}{1-p}\right) + h^o$ and thus, $h(0) = \log\left(\frac{p}{1-p}\right) + h^o(0) \neq 0$ in general.

parameters at the same time since p appears separately from $p(Z)$ in the likelihood function. Considering these advantages of ML, one may want to use ML instead of the conditional ML. However, we note that the kind of transformation in the likelihood function of (13) is not valid if $f_{(st)}(\cdot)$ and $f_{(we)}(\cdot)$ depend on some model parameters. If this is the case, the full ML requires to estimate $f_{(st)}(\cdot)$ and $f_{(we)}(\cdot)$ together with other model parameters, which we want to avoid[23]. For this reason, we will focus on the conditional ML. Though our proposed sieve conditional ML estimator do not achieve the efficiency, we note that the asymptotic results of the sieve ML estimator in the literature will be identically applied to the sieve conditional ML. The results obtained for the conditional ML in the following subsections can be applied to the ML estimator $\widetilde{\alpha}^o_{Full}$ in a similar manner.

5.2.1 Consistency and Convergence Rates of the Sieve Conditional ML

The consistency of sieve MLE was derived in Wong and Severini (1991) and Geman and Hwang (1992) for i.i.d data. Some consistency results of sieve M-estimators can be found in Gallant (1987) and Gallant and Nychka (1987). Chen (2005) presented a consistency result of sieve extremum estimators allowing for non-compact infinite-dimensional $\mathcal{A}$, which is an extension of Theorem 2.1 in Newey and McFadden (1994) and of Lemma A1 in Newey and Powell (2003). Using Theorem 3.1 in Chen (2005), we establish the consistency under a pseudo metric $\|\cdot\|_s$ defined by

$$\|\alpha_1 - \alpha_2\|_s = \|\theta_1 - \theta_2\|_E + \|h_1 - h_2\|_\infty$$

where $\|\cdot\|_E$ is the Euclidean norm and $\|h\|_\infty = \sup_{z \in \mathbb{S}(Z)} |h(z)|$. We make additional assumptions.

Assumption 5.3 *(SA-4)*
(i) $\{Y_{1i}, Y_{2i}, W_i\}_{i=1}^n$ are iid; (ii) $\alpha_0 = (\theta_0, h_0) \in \mathcal{A} \equiv \Theta \times \mathcal{H}$; (iii) Θ is compact with nonempty interior and $\mathcal{H} = \Lambda^{\nu_1}_{C_1}(\mathbb{S}(Z))$; (iv) $K_n \to \infty$ and $K_n/n \to 0$.

Lemma 5.2 *(Consistency)*
*Suppose Assumptions **SA-2**, **SA-3**, and **SA-4** hold and suppose Condition 9 (Lipschitz) in the Appendix holds. Then, $\|\widehat{\alpha}_n - \alpha_0\|_s = o_p(1)$.*

The proof can be found in the Appendix. This consistency result is obtained combining the identification condition and the uniform convergence of the criterion function as in the case of the parametric extremum estimation.

Now we consider the convergence rate of the sieve conditional ML estimator under a weaker metric[24]. We present a convergence rate using Theorem 3.2 of Chen (2005) which is a version of Chen and Shen (1996) for i.i.d data. We redefine, $\widehat{\alpha}_n$, the approximate sieve ML that satisfies

$$\widehat{Q}(\widehat{\alpha}_n) \geq \sup_{\alpha \in \mathcal{A}_n} \widehat{Q}(\alpha) - O_p\left(\varepsilon_n^2\right), \text{ with } \varepsilon_n = o(1). \tag{14}$$

Now suppose that $\mathcal{A} = \Theta \times \mathcal{H}$ is convex in α_0 so that $\alpha_0 + \tau(\alpha - \alpha_0) \in \mathcal{A}$ for all small $\tau \in [0, 1]$ and all fixed $\alpha \in \mathcal{A}$. Suppose that the pathwise derivative of $l(\cdot)$ at the direction $[\alpha - \alpha_0]$ is well-defined for almost all $w \times y$ in the support of $\mathbb{S}(W) \times \mathbb{S}(Y)$. We denote the pathwise first derivative at the direction $[\alpha - \alpha_0]$

---

[23] A semiparametric estimation of a mixing distribution with nonparametric mixing components has not been established yet in the literature except some identification result in Kitamura (2004).

[24] Shen and Wong (1994), and Birgé and Massart (1998) derived the rates for general sieve M-estimation. Van de Geer (1993) and Wong and Shen (1995) presented the rates for sieve MLE for i.i.d data.

evaluated at α_0 by

$$
\begin{aligned}
\frac{dl(y_i|w_i, \alpha_0)}{d\alpha}[\alpha - \alpha_0] &\equiv \lim_{\tau \to 0} \frac{dl(y_i|w_i, (1-\tau)\alpha_0 + \tau\alpha)}{d\tau} \\
&= \frac{\partial l(y_i|w_i, \alpha_0)}{\partial \theta'}(\theta - \theta_0) + \frac{dl(y_i|w_i, \alpha_0)}{dh}[h - h_0].
\end{aligned}
$$

Now we define the $L_2(P_0)$-norm, $\|\alpha - \alpha_0\|_2$, based on the pathwise derivative of $l(\cdot)$ evaluated at α_0, i.e.

$$
\|\alpha - \alpha_0\|_2 = \sqrt{E\left[\left(\frac{dl(Y_i|W_i, \alpha_0)}{d\alpha}[\alpha - \alpha_0]\right)^2\right]}. \tag{15}
$$

This[25] is the ML version of Ai and Chen (2003)'s $L_2(P_0)$-metric and the conditional ML version of the metric used by Wong and Severini (1991).

Proposition 5.1 *Let $\widehat{\alpha}_n$ be the approximate sieve ML defined in (14). Suppose Assumptions **SA-2**, **SA-3**, and **SA-4** hold and suppose Conditions 10-12 (Lipschitz) in the Appendix hold. Then, we have*

$$
\|\widehat{\alpha}_n - \alpha_0\|_2 = O_p\left(\max\left\{O\left(\sqrt{\frac{K_n}{n}}\right), O\left(K_n^{-\nu_1/d_z}\right)\right\}\right).
$$

Moreover, with $K_n = n^{1/(2\nu_1/d_z + 1)}$ and $\nu_1/d_z > 1/2$, we have $\|\widehat{\alpha}_n - \alpha_0\|_2 = O_p\left(n^{-\frac{\nu_1/d_z}{2\nu_1/d_z + 1}}\right)$.

The proof of this proposition can be found in the Appendix.

5.2.2 Asymptotic Normality

In this section, we derive the $\sqrt{n}$-asymptotic normality[26] of the structural parameters $\widehat{\theta}_n$. The following discussion and notations are based on Theorem 4.3 of Chen (2005), which is a simplified version of Shen (1997) and Chen and Shen (1998).

Suppose the functional of interest, $f : \mathcal{A} \to \mathbb{R}$, is smooth in the sense that

$$
\frac{df(\alpha_0)}{d\alpha}[\alpha - \alpha_0] \equiv \lim_{\tau \to 0} \frac{f(\alpha_0 + \tau(\alpha - \alpha_0)) - f(\alpha_0)}{\tau}
$$

is well defined and

$$
\left\|\frac{df(\alpha_0)}{d\alpha}\right\| \equiv \sup_{\alpha \in \mathcal{A}, \|\alpha - \alpha_0\|_2 > 0} \frac{\left|\frac{df(\alpha_0)}{d\alpha}[\alpha - \alpha_0]\right|}{\|\alpha - \alpha_0\|_2} < \infty.
$$

Now let $\overline{V}$ denote the closure of the linear span of $\mathcal{A} - \alpha_0$ under the metric $\|\alpha - \alpha_0\|_2$. Then, $\left(\overline{V}, \|\cdot\|_2\right)$ is a Hilbert space with inner product

$$
\langle v_1, v_2 \rangle = E\left[\left(\frac{dl(Y_i|W_i, \alpha_0)}{d\alpha}[v_1]\right)\left(\frac{dl(Y_i|W_i, \alpha_0)}{d\alpha}[v_2]\right)\right].
$$

---

[25] For the ML context, this norm is very natural since it is the Fisher norm (see Wong and Severini (1991)).

[26] Wong and Severini (1991) established the $\sqrt{n}$-asymptotic normality and efficiency of smooth functionals of nonparametric MLE with parameter space $\mathcal{A}_n \equiv \mathcal{A} = \Theta \times \mathcal{H}$. Shen (1997) extended their results to sieve MLE allowing for highly curved (nonlinear) least favorable directions while Chen and Shen (1998) extended Shen (1997) to general sieve M-estimation with stationary weakly dependent data.

Then, by the Riesz representation theorem, there exists $v^* \in \overline{V}$ such that, for any $\alpha \in \mathcal{A}$,

$$\frac{df(\alpha_0)}{d\alpha}[\alpha - \alpha_0] = \langle \alpha - \alpha_0, v^* \rangle \text{ iff } \left\| \frac{df(\alpha_0)}{d\alpha} \right\| < \infty. \tag{16}$$

In particular, we let $f(\alpha) = \lambda'\theta$ for any fixed and nonzero $\lambda \in \mathbb{R}^{d_\theta}$. Then, $f(\alpha) \equiv \lambda'\theta$ is a linear functional on $\overline{V}$. To estimate $f(\alpha) \equiv \lambda'\theta$ at a $\sqrt{n}$ rate, $f(\alpha)$ has to be bounded ($\sup_{0 \neq \alpha - \alpha_0 \in \overline{V}} |f(\alpha) - f(\alpha_0)| / \|\alpha - \alpha_0\|_2 < \infty$) according to Van der Vaart (1991) and Shen (1997).

Now let $\overline{\mathcal{H}} - h_0$ denote the closure of the linear span of $\mathcal{H} - h_0$. Define $b_j^* \in \overline{\mathcal{B}} \equiv \overline{\mathcal{H}} - h_0$ for each component θ_j of θ such that

$$b_j^* = \underset{b_j \in \overline{\mathcal{B}}}{\operatorname{argmin}} E\left[\left(\frac{dl(Y|W,\alpha_0)}{d\theta_j} - \frac{dl(Y|W,\alpha_0)}{dh}[b_j] \right)^2 \right]. \tag{17}$$

Now define $b^* = (b_1^*, \ldots, b_{d_\theta}^*)$,

$$\frac{dl(Y|W,\alpha_0)}{dh}[b^*] = \left(\frac{dl(Y|W,\alpha_0)}{dh}[b_1^*], \ldots, \frac{dl(Y|W,\alpha_0)}{dh}[b_{d_\theta}^*] \right),$$

and $D_{b^*}(Y,W) \equiv \frac{dl(Y|W,\alpha_0)}{d\theta'} - \frac{dl(Y|W,\alpha_0)}{dh}[b^*]$. We impose

Assumption 5.4 (SA-5)
(i) $\theta_0 \in int(\Theta)$; (ii) $E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right]$ is positive definite; (iii) each element $b_j^*(Z)$ belongs to the Hölder space $\Lambda_{C_j}^{m_j}(\mathbb{S}(Z))$ with $m_j > d_z/2$.

Now note that $\frac{df(\alpha_0)}{d\alpha}[\alpha - \alpha_0] = (\theta - \theta_0)'\lambda$ which implies that

$$f(\alpha) - f(\alpha_0) - \frac{df(\alpha_0)}{d\alpha}[\alpha - \alpha_0] = 0. \tag{18}$$

In addition, we can show that for $f(\alpha) \equiv \lambda'\theta$ with $\lambda \in \mathbb{R}^{d_\theta}, \lambda \neq 0$,

$$\sup_{0 \neq \alpha - \alpha_0 \in \overline{V}} \frac{|f(\alpha) - f(\alpha_0)|^2}{\|\alpha - \alpha_0\|_2^2} = \lambda' \left(E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right] \right)^{-1} \lambda$$

which implies $f(\alpha) = \lambda'\theta$ is bounded (in the sense of $\sup_{0 \neq \alpha - \alpha_0 \in \overline{V}} |f(\alpha) - f(\alpha_0)| / \|\alpha - \alpha_0\|_2 < \infty$) if and only if $E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right]$ is positive-definite (Assumption **SA-5** (ii)). For this case, there exists $v^* \in \overline{V}$ such that

$$f(\alpha) - f(\alpha_0) \equiv \lambda'(\theta - \theta_0) = \langle v^*, \alpha - \alpha_0 \rangle \text{ for all } \alpha \in \mathcal{A} \tag{19}$$

by (18) and by the Riesz representation theorem (see (16)). We find that $v^* \equiv (v_\theta^*, v_h^*) \in \overline{V}$ satisfies (19) with $v_\theta^* = \left(E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right] \right)^{-1} \lambda$ and $v_h^* = -b^* \times v_\theta^*$.

The following theorem states that we can achieve the $\sqrt{n}$-asymptotic normality for the structural parameters.

Theorem 5.1 *Suppose Assumptions **SA-2**, **SA-3**, **SA-4**, and **SA-5** hold and suppose Conditions 10-12 (Lipschitz) in the Appendix hold. Then, we have*

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \underset{d}{\to} N(0, \Omega_*)$$

where $\Omega_* = \left\{ E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right] \right\}^{-1}$.

The proof of this theorem can be found in the Appendix.

5.2.3 A Consistent Covariance Estimator

To do a statistical inference of the structural parameters based on Theorem 5.1, we need a consistent estimator of Ω_*. First, we need a consistent estimator of b^*. We estimate b_j^* by $\widehat{b}_j^*$, $j = 1, \ldots, d_\theta$ such that

$$\widehat{b}_j^* = \operatorname{argmin}_{b_j \in \mathcal{H}_n} \sum_{i=1}^n \left(\frac{dl(y_i|w_i, \widehat{\alpha}_n)}{d\theta_j} - \frac{dl(y_i|w_i, \widehat{\alpha}_n)}{dh}[b_j] \right)^2. \tag{20}$$

Note that for linear sieves, $\widehat{b}_j^*$ is easily obtained by regressing the derivatives of $l(\cdot|\cdot, \widehat{\alpha}_n)$ with respect to θ_j on the derivatives of $l(\cdot|\cdot, \widehat{\alpha}_n)$ with respect to h. Finally, we estimate Ω_* by

$$\widehat{\Omega}_* \equiv \left(\frac{1}{n} \sum_{i=1}^n D_{\widehat{b}^*}(y_i, w_i, \widehat{\alpha}_n)' D_{\widehat{b}^*}(y_i, w_i, \widehat{\alpha}_n) \right)^{-1}$$

where $\widehat{b}^* = (\widehat{b}_1^*, \ldots, \widehat{b}_{d_\theta}^*)$ and $D_{b^*}(Y, W, \alpha) = \frac{\partial l(Y|W, \alpha)}{\partial \theta'} - \frac{dl(Y|W, \alpha)}{dh}[b^*]$. We note that $\widehat{\Omega}_*$ is consistent under suitable conditions.

Proposition 5.2 *Suppose Assumptions **SA-2**, **SA-3**, **SA-4**, and **SA-5** hold and suppose Conditions 10-12 (Lipschitz) in the Appendix hold. Then, $\widehat{\Omega}_* = \Omega_* + o_p(1)$.*

5.2.4 Alternative Sieve Minimum Distance Estimator (SMD)

Under uniqueness of equilibrium, the true conditional probabilities of observed outcomes should be identical with the conditional probabilities implied by a model evaluated at the true parameter. This implies that the game models can be represented by a set of conditional moment conditions. To be precise, we have

$$\begin{aligned}
0 &= E\left[Y_1 Y_2 - \Pr(Y_1 = 1, Y_2 = 1|W, \alpha_0)|W\right] \equiv E\left[\rho_1(Y_1, Y_2, W, \theta_0, h_0)|W\right] \\
0 &= E\left[Y_1(Y_2 - 1) - \Pr(Y_1 = 1, Y_2 = 0|W, \alpha_0)|W\right] \equiv E\left[\rho_2(Y_1, Y_2, W, \theta_0, h_0)|W\right] \\
0 &= E\left[(Y_1 - 1)Y_2 - \Pr(Y_1 = 0, Y_2 = 1|W, \alpha_0)|W\right] \equiv E\left[\rho_3(Y_1, Y_2, W, \theta_0, h_0)|W\right].
\end{aligned} \tag{21}$$

Based on these conditional moment restrictions, we can estimate the parameters of interest using the sieve minimum distance estimation proposed by Ai and Chen (2003) as an alternative to the sieve conditional ML. Note that only three outcomes are independent out of the four possible outcomes of (Y_1, Y_2) since the sum of probabilities of four outcomes is always one. Now let $\rho(\cdot) = (\rho_1(\cdot), \rho_2(\cdot), \rho_3(\cdot))'$ and then

$$E\left[\rho(Y, W, \theta_0, h_0)|W\right] = 0. \tag{22}$$

Ai and Chen (2003) provides a general framework dealing with an estimation for models of conditional moment restrictions such as (22) and obtain the consistency and $\sqrt{n}$-asymptotic normality of the estimator for θ_0.

Now let $m(W, \alpha)$ be the conditional mean function of the moment equation $E\left[\rho(Y_1, Y_2, W, \alpha)|W\right]$ and denote by $\widehat{m}(W, \alpha)$, a linear sieve estimator of $m(\cdot, \cdot)$ as

$$\widehat{m}_l(W, \alpha) = \sum_{j=1}^n \rho_l(Y_j, W_j, \alpha) p^{k_n}(W_j)'(P'P)^{-1} p^{k_n}(W), \quad l = 1, \ldots, \dim(\rho) = 3,$$

where $p^{k_n}(W)$ is a tensor-product sieve (such as power series, Fourier series, or B-splines) and $P = (p^{k_n}(W_1), \ldots, p^{k_n}(W_n))'$. Also denote by $\widehat{\Sigma}(W)$, a consistent estimator of a positive definite matrix $\Sigma(W)$

for any given W. Then, an SMD estimator based on the conditional moment restriction of $(22)^{27}$ is obtained by

$$\widehat{\alpha}_{MD} = (\widehat{\theta}_{MD}, \widehat{h}_{MD}) = \operatorname{argmin}_{\alpha=(\theta,h) \in A_n \equiv \Theta \times \mathcal{H}_n} \frac{1}{n} \sum_{i=1}^{n} \widehat{m}(W_i, \alpha)' [\widehat{\Sigma}(W_i)]^{-1} \widehat{m}(W_i, \alpha) \qquad (23)$$

where $\mathcal{H}_n$ is the sieve space defined in $(8)^{28}$.

5.3 Estimation of the Type Distribution

In Section 5.2, we argue that we can identify the parameter p from $p(Z)$ by (6). From this result, we propose an estimator for p as

$$\widehat{p}_n = \frac{1}{n} \sum_{i=1}^{n} L(\widehat{h}_n(z_i)) = \frac{1}{n} \sum_{i=1}^{n} \frac{\exp(\widehat{h}_n(z_i))}{1 + \exp(\widehat{h}_n(z_i))} \qquad (24)$$

where $\widehat{h}_n(\cdot)$ is obtained from (10) or (23). Applying the mean value theorem with $\widetilde{h}_n$ that lies between $\widehat{h}_n$ and h_0, we have

$$
\begin{aligned}
\widehat{p}_n - p_0 &= \frac{1}{n} \sum_{i=1}^{n} \left(L(\widehat{h}_n(Z_i)) - L(h_0(Z_i)) \right) + \frac{1}{n} \sum_{i=1}^{n} \left(L(h_0(Z_i)) - E\left[L(h_0(Z_i)) \right] \right) \qquad (25) \\
&= \frac{1}{n} \sum_{i=1}^{n} L(\widetilde{h}_n(Z_i))(1 - L(\widetilde{h}_n(Z_i)))(\widehat{h}_n(Z_i) - h_0(Z_i)) + \frac{1}{n} \sum_{i=1}^{n} \left(L(h_0(Z_i)) - E\left[L(h_0(Z_i)) \right] \right) \\
&\leq \frac{1}{4} \left\| \widehat{h}_n - h_0 \right\|_{\infty} + o_p(1)
\end{aligned}
$$

where the second equality is obtained by $L'(\cdot) = L(\cdot)(1 - L(\cdot))$ and the last result holds since $L(\cdot)(1 - L(\cdot)) \leq \frac{1}{4}$ uniformly and the second term in the RHS of (25) is $o_p(1)$ by LLN noting $|L(h_0(\cdot))| < 1$ uniformly and $\{Z_i\}_{i=1}^{n}$ are iid.

Thus, $\widehat{p}_n$ is consistent as long as $\widehat{h}_n$ is consistent. Now we can derive the asymptotic distribution of $\widehat{p}_n$ similarly with Chen, Linton, and van Keilegom (2003) since (6) can be written as a moment condition

$$m(p, h) = p - L(h), \ E[m(p_0, h_0)] = 0$$

and since we have an initial estimator $\widehat{h}_n$ such that $\left\| \widehat{h}_n - h_0 \right\|_{\infty} = o_p(n^{-1/4})$ from the sieve conditional ML or the SMD^{29}. We obtain the following result. We let $M(h) = \int_{\mathbb{S}(Z)} L(h) dF_Z$ and $M_n(h) = \frac{1}{n} \sum_{i=1}^{n} L(h(Z_i))$.

Proposition 5.3 *Suppose (i)* $\left\| \widehat{h}_n - h_0 \right\|_{\infty} = o_p(n^{-1/4})$ *and (ii)*

$$\sqrt{n} \left(\int_{\mathbb{S}(Z)} L(h_0)(1 - L(h_0)) \left(\widehat{h}_n - h_0 \right) dF_Z + M_n(h_0) - M(h_0) \right) \xrightarrow{d} N(0, V_p). \qquad (26)$$

---

[27] Alternatively, the conditional moment conditions of (22) can be estimated using a penalized empirical likelihood estimation (PELE) proposed by Otsu (2003). The main difference between these two estimators is that in the PELE, we do not restrict the class of functions $\mathcal{H}$ to be compact and do not approximate the function space $\mathcal{H}$ with sieves $\mathcal{H}_n$. Instead, the maximization is taken over the function space $\mathcal{H}$ while controlling the physical plausibility (such as roughness) of functions using a penalty function. Otsu (2003) shows that the optimally weighted kernel smoothed version of MD (the original version of the SMD considered in Ai and Chen (2003)'s working paper version) and the PELE are asymptotically equivalent in the first order sense. From this result, we expect that the PELE is first order asymptotically equivalent to the optimally weighted SMD. However, we conjecture that the PELE is preferable to the SMD in terms of higher order asymptotics from the relationship between parametric GMM and EL estimators (see Newey and Smith (2004)).

[28] The asymptotic property of this estimator is provided in the dissertation version of this paper.

[29] Combining with the result of Lemma 2 in Chen and Shen (1998), which shows that $\|h - h_0\|_{\infty} \leq \|h - h_0\|_2^{\frac{2\nu_1}{2\nu_1 + d_z}}$.

Then, $\sqrt{n}\left(\widehat{p}_n - p_0\right) \underset{d}{\rightarrow} N(0, V_p)$.

Note that the first term in the LHS of (26) appears due to the fact that we use an initial estimator $\widehat{h}_n$ and will disappear if $\widehat{h}_n = h_0$. The proof of this Proposition can be found in the Appendix. In the Appendix, we also show that the condition (ii) of Proposition 5.3 holds for the sieve conditional ML estimator under Assumptions in Theorem 5.1. We note that even if an explicit form of V_p can be derived, a feasible estimator of V_p may be difficult to calculate. Alternatively, we can use the ordinary nonparametric bootstrap. The following proposition shows that we can consistently estimate the distribution of $\sqrt{n}\left(\widehat{p}_n - p_0\right)$ using the bootstrap under some conditions. We use "$*$" to denote the bootstrap counterpart of the original sample $\{Z_i\}_{i=1}^n$. We let $M_n^*(h) = \frac{1}{n}\sum_{i=1}^n L(h(Z_i^*))$.

Proposition 5.4 *Suppose (i) with P^*-probability tending to one, $\widehat{h}_n^* \in \mathcal{H}$, and $\left\|\widehat{h}_n^* - \widehat{h}_n\right\|_\infty = o_{P^*}(n^{-1/4})$;(ii)* $\left\|\widehat{h}_n - h_0\right\|_\infty = o(n^{-1/4})$ *a.s.;(iii) for any positive sequence $\delta_n = o(1)$,*

$$\sup_{\|h - h_0\|_\infty \le \delta_n} |M_n(h) - M(h) - M_n(h_0) + M(h_0)| = o(n^{-1/2}) \text{ a.s.;}$$

(iv) $\sqrt{n}\left(\int_{\mathbb{S}(Z)} L(\widehat{h}_n)(1 - L(\widehat{h}_n))\left(\widehat{h}_n^* - \widehat{h}_n\right) dF_Z + M_n^*(\widehat{h}_n) - M_n(\widehat{h}_n)\right) = N(0, V_p) + o_{P^*}(1).$

Then, $\sqrt{n}\left(\widehat{p}_n^* - \widehat{p}_n\right)$ *converges in distribution to $N(0, V_p)$ in P^*-probability.*

6 Discussion: Set Identification

In previous sections, to achieve uniqueness of equilibrium, we have used an equilibrium refinement developed by Cho and Kreps (1987). For signaling games, however, the "self-fulfilling property" is essential, which can yield multiple equilibria depending on different beliefs on plays of the other party. In this section, we will consider the possibility of multiple equilibria under PBE without attempting to resolve the equilibrium selection problem. As a consequence, we give up point identification since we do not have a well-defined likelihood function and adopt a set identification approach. Here we consider the model where some asymptotic inequalities may define a region of parameters rather than a single point in the parameter space. The idea is as follows. When multiple equilibria arise, there are regions of unobservables that are consistent with necessary conditions for more than one equilibrium. Therefore, the probability of an event, implied by the necessary condition, is greater than or equal to the true probability of the event. As a consequence, these necessary conditions will provide a set of inequality constraints on the parameters rather than a set of equality conditions. Interestingly, we note that as suggested in Figure 3 compared to Figure 4, the multiplicity of equilibria may help the identification of parameters (though they are set-identified) since there are more variations of equilibria.

We can adopt a simple estimation strategy that utilizes the sample analog of these population necessary conditions as in Chernozhukov, Hong, and Tamer (2003) and Andrews, Berry, and Jia (2004). The approach we take is from Andrews, Berry, and Jia (2004). Noting that Andrews, Berry, and Jia (2004) do not allow for infinite dimensional parameters[30], we consider a parameterization of $h(\cdot)$ for the game with public signals. A separate study of Kim (2006) illustrates that we can actually allow for infinite dimensional parameters in this set estimation and inference for a fully symmetric version of the signaling game. Now let $h(\cdot) = h(\cdot, \delta_0)$ for

---

[30] It is because ABJ utilizes finite numbers of cells to facilitate the estimation, which is not compatible with infinite dimensional parameters. Simply it violates the order condition for identification.

$\delta_0 \in \mathcal{D}$ and let $\vartheta_0 = (\theta_0', \delta_0)'$. Recall $Y_i = (Y_{1i}, Y_{2i})$, $y = (y_1, y_2)$, and $\varepsilon_i = (\varepsilon_{1i}, \varepsilon_{2i})$. Denote $\Omega(Y = y, W, \vartheta_0)$ to be the region of ε under which Y takes the value y given W and ϑ_0. To be precise,

$$\Omega(y, w, \vartheta_0) \equiv \Omega(Y = y, W = w, \vartheta_0) = \{\varepsilon | Y_1 = y_1, Y_2 = y_2 \text{ given } W = w \text{ and } \vartheta_0\}.$$

Then, the probability that the necessary conditions for $Y = y$ holds will equal the probability that ε belongs to $\Omega(y, w, \vartheta_0)$ given $W = w$ and ϑ_0. The sieve ML or SMD approaches taken in the previous sections typically proceed by identifying a one-to-one mapping between the possible discrete outcomes and regions of the unobservables. Under the existence of multiple equilibria, probabilities of observed events are not necessarily equal to the probabilities of the associated regions of unobservables. Now for any $(y, w) \in \mathbb{S}(Y) \times \mathbb{S}(W)$, the probability is defined to be

$$P(y|w, \vartheta) = \Pr(\varepsilon \in \Omega(y, w, \vartheta)) \tag{27}$$

When $\vartheta = \vartheta_0$, this is a simple ε-orthant probability. Note that at the true parameter value $\vartheta = \vartheta_0$, the probabilities of the necessary conditions must be at least as large as the true probabilities of the events $y \in \mathbb{S}(Y)$ given $W = w$, denoted by $P_0(y|w)$:

$$P(y|w, \vartheta_0) \geq P_0(y|w), \ \forall (y, w) \in \mathbb{S}(Y) \times \mathbb{S}(W). \tag{28}$$

Notice that this inequality follows from the fact that the outcome y implies the necessary conditions for y but the necessary condition need not imply the outcome y. Now based on the population inequality conditions of (28), we can follow Andrews, Berry, and Jia (2004)'s approach.

From the results of Section 3 (Figure 3), it is not difficult to construct the model probabilities for each discrete outcome of the game using (27). We note that though the usage of a mixed strategy by players is well justified theoretically, it is still a matter of question whether a mixed strategy is used in the real world. Some experimental[31] and empirical[32] studies are trying to answer this question but findings are mixed. As an attractive alternative, one may proceed the analysis assuming that any observed outcome can arise without any restriction in the regions where we have semi-separating equilibria in the game. This may still provide a tight bound for the estimate of the parameters. We may compare this result with that of allowing the semi-separating equilibria.

6.1 Set Estimator

The model probabilities $\{P(y|W_i, \vartheta) : i = 1, \ldots, n\}$ induced by the games we have studied have analytic closed form expressions. This makes our problem simpler since we do not need to consider the simulation of the probabilities. However, we still need to consider the construction of W cells to achieve a reliable estimation. By construction, we have $\mathbb{S}(Y) = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$. To make notation simple while allowing possible extensions to general cases, we let $y_1 = (1, 1)$, $y_2 = (1, 0)$, $y_3 = (0, 1)$, and $y_4 = (0, 0)$. We also let $J_Y = 4$. Then, we have $\mathbb{S}(Y) = \{y_j : j = 1, \ldots, J_Y\}$.

Now we briefly review the data-dependent construction of W cells following Andrews, Berry, and Jia (2004). Consider a set $\{q_\gamma : \gamma \in \Gamma\}$ of real-valued weight functions on the support $\mathbb{S}(W)$ of W_i, where γ is a subset of $\mathbb{S}(W)$ and Γ is a collection of subsets of $\mathbb{S}(W)$. In particular, for each y_j, we consider $\mathcal{M}_j$ subsets of $\mathbb{S}(W)$ denoted by $\gamma_{j,m}$:

$$\begin{aligned} \Gamma &= \{\gamma_{j,m} \subset \mathbb{S}(W) : (j, m) \in \mathcal{I}_{J_Y, \mathcal{M}}\}, \text{ where} \\ \mathcal{I}_{J_Y, \mathcal{M}} &= \{(j, m) : m = 1, \ldots, \mathcal{M}_j, j = 1, \ldots, J_Y\}. \end{aligned} \tag{29}$$

---

[31] See O'Neill (1987), Rapoport and Boebel (1992), Mookherjee and Sopher (1994), Ochs (1995), and McCabe et al. (2000).
[32] See Walker and Wooders (2001) and Chiappori, Levitt, and Groseclose (2002).

The functions $\{q_\gamma : \gamma \in \Gamma\}$ aggregate and/or weight the necessary conditions for an equilibrium over different values of w. Now let

$$\widehat{\Gamma}_n = \left\{\widehat{\gamma}_{n,j,m} \subset \mathbb{S}(W) : (j,m) \in \mathcal{I}_{J_Y,\mathcal{M}}\right\},$$

where $\widehat{\gamma}_{n,j,m}$ is a random subset of $\mathbb{S}(W)$. For the consistency of the set estimator, we require that $\widehat{\Gamma}_n \underset{p}{\to} \Gamma_0$ under certain metric where

$$\Gamma_0 = \left\{\gamma_{0,j,m} \subset \mathbb{S}(W) : (j,m) \in \mathcal{I}_{J_Y,\mathcal{M}}\right\}.$$

The set estimator proposed by Andrews, Berry, and Jia (2004) is obtained as follows. Define

$$
\begin{aligned}
c_0(j,\gamma,\vartheta) &= \int \left(P(y_j|w,\vartheta) - P_0(y_j|w)\right) q_\gamma(w) dF_W(w) \text{ and} &(30)\\
\widehat{c}_n(j,\gamma,\vartheta) &= n^{-1}\sum_{i=1}^n \left(P(y_j|W_i,\vartheta) - \mathbf{1}[Y_i = y_j]\right) q_\gamma(W_i).
\end{aligned}
$$

Necessary conditions for ϑ to be the true parameters are

$$P(y|w,\vartheta) - P_0(y|w) \geq 0, \; \forall (y,w) \in \mathbb{S}(Y) \times \mathbb{S}(W) \tag{31}$$

which implies that

$$c_0(j,\gamma_{0,k,m},\vartheta) \geq 0, \; \forall (j,m) \in \mathcal{I}_{J_Y,\mathcal{M}}. \tag{32}$$

Define

$$\Theta_0 = \{\vartheta \in \Theta \times \mathcal{D} : (31) \text{ holds}\} \text{ and } \Theta_+ = \{\vartheta \in \Theta \times \mathcal{D} : (32) \text{ holds}\}.$$

By definition, the set Θ_0 is the smallest set of parameter values that necessarily includes the true value ϑ_0. By construction, $\Theta_+ \supset \Theta_0$ since (31) implies (32). Now the set estimator is obtained as

$$\widehat{\Theta}_n = \{\vartheta \in \Theta \times \mathcal{D} : \vartheta \text{ minimizes } Q_n(\vartheta) \text{ over } \Theta \times \mathcal{D}\}$$

where

$$Q_n(\vartheta) = \sum_{(j,m)\in\mathcal{I}_{J_Y,\mathcal{M}}} \left|\widehat{c}_n(j,\widehat{\gamma}_{n,j,m},\vartheta)\right| \cdot \mathbf{1}[\widehat{c}_n(j,\widehat{\gamma}_{n,j,m},\vartheta) \leq 0]. \tag{33}$$

Provided that Assumptions 1-6 of Andrews, Berry, and Jia (2004) are satisfied, we have

$$d(\widehat{\Theta}_n,\Theta_+) \underset{p}{\to} 0$$

where $d(\cdot,\cdot)$ is the Hausdorff metric that measures the distance between two sets. For the inference of this set estimator, Andrews, Berry, and Jia (2004) provide confidence intervals for individual parameters and confidence regions, whose critical values are obtained using the bootstrap methods.

6.2   Set Estimation of the Type Distribution

We have noted that in the conditional probabilities of observed outcomes, implied by the model, p does not appear separately from $p(Z)$. However, from (6), we can still identify the type distribution parameter p. Using the relationship in (6), we obtain a set estimator of p_0 such that

$$\widehat{\mathcal{P}}_n = \left\{p : p = \frac{1}{n}\sum_{i=1}^n L(h(z_i,\delta)) = \frac{1}{n}\sum_{i=1}^n \frac{\exp(h(z_i,\delta))}{1 + \exp(h(z_i,\delta))} \text{ for each } \delta \text{ in } \widehat{\Theta}_n\right\}$$

We note that as long as $d(\widehat{\Theta}_n,\Theta_+) = o_p(1)$, $\widehat{\mathcal{P}}_n$ converges to its population counterpart $\mathcal{P}_+$ defined by

$$\mathcal{P}_+ = \left\{p : p = E\left[L(h)\right] = E\left[\frac{\exp(h(Z,\delta))}{1 + \exp(h(Z,\delta))}\right] \text{ for each } \delta \text{ in } \Theta_+\right\}.$$

Proposition 6.1 *Suppose* $d\left(\widehat{\Theta}_n, \Theta_+\right) = o_p(1)$ *and* $\sup_{\delta \in \mathcal{D}, z \in \mathbb{S}(Z)} \left\| \frac{\partial h(z, \pi)}{\partial \pi'} \right\|_E < \infty.$
Then, $d\left(\widehat{\mathcal{P}}_n, \mathcal{P}_+\right) = o_p(1).$

7 Concluding Remarks

This paper develops an econometric modeling of a signaling game with two players where one player, *informed party*, has private information summarized by types. In particular, we provide an estimation strategy that identifies the payoffs structure and the distribution of types from data of observed actions. Though multiplicity of equilibria arises in the game, we show that uniqueness of equilibrium given a realization of payoffs can be achieved as long as players play a PBE and choose only one equilibrium out of multiple equilibria using the refinement of Cho and Kreps (1987). This uniqueness enables us to derive well-defined conditional probabilities that are useful for the estimation.

To provide some empirical relevance, we consider public signals about the type of a player. Technically, these signals cannot be manipulated by the *informed* player or at least the player has no incentive to manipulate the signals. Therefore, the *uninformed party* will use this information to update her belief on types after observing an action of the *informed party* when a pooling equilibrium is played. Since the mixing distribution of these non-strategic signals on types is nonparametrically specified, we estimate the model using a sieve conditional MLE where the infinite dimensional parameters are approximated by sieves. Noting that the conditional probability of choosing a certain combination of actions can be written in terms of several conditional moment restrictions, as an alternative, we estimate the model using the sieve minimum distance (MD) estimation. In both methods, we obtain the consistency and the root n-asymptotic normality of structural parameters estimates.

We note that in the signaling game, multiple equilibria naturally arise given a realization of payoffs due to the *self-fulfilling* property. We resolve this problem by refining the equilibrium using an equilibrium selection rule that may be arbitrary and cannot be justified. As an attractive alternative to the previous approach, we allow for the possibility of multiple equilibria, without attempting to resolve the equilibrium selection problem. As a consequence, we give up point identification since we do not have a well-behaved likelihood function and adopt the set identification approach. In particular, we consider the model where some asymptotic inequalities may define a region of parameters rather than a single point in the parameter space. We adopt the important work of Andrews, Berry, and Jia (2004) for the set estimation.

Appendix I

A   Equilibrium Refinement and Uniqueness of Equilibrium (Proof of Theorem 3.2)

- Under region $\boldsymbol{A}_a \equiv \{(\varepsilon_1, \varepsilon_2) \, | \, \varepsilon_1 > \mu + X_1'\beta_1 + \phi_{1w}$ and $X_2'\beta_2 + (\phi_{2s} + \phi_{2w})p - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$, we show that the pooling with (B, B) cannot survive the Intuitive Criterion of Cho and Kreps (1987) while the pooling with (Q, Q) survives it.

    - Pooling with $(A_{1t_s}, A_{1t_w}) = (B, B)$ with $A_{2|B} = F$

    Note $u_1(t_s; B, F) > u_1(t_s; Q, NF)$ since $\mu + X_1'\beta_1 - \varepsilon_1 - \phi_{1s} < 0$ under $\boldsymbol{A}_a$ and note $u_1(t_s; B, F) > u_1(t_s; Q, F)$ since $\phi_{1s} > 0$. This means that the *strong* type has no incentive to deviate in any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|Q) = 0$ and hence Player 2 will choose F after observing the deviation play Q under region $\boldsymbol{A}_a$. Now we need to check whether the *weak* type is better off by deviation under this situation. Note
    $$u_1(t_w; B, F) = -\phi_{1w} < u_1(t_w; Q, F) = 0.$$

    Thus, the *weak* type will deviate for sure. Therefore, the equilibrium outcome fails the Intuitive Criterion.

    - Pooling with $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ with $A_{2|Q} = F$

    Note $u_1(t_w; Q, F) > u_1(t_w; B, NF)$ since $\mu + X_1'\beta_1 - \varepsilon_1 - \phi_{1w} < 0$ under $\boldsymbol{A}_a$ and note $u_1(t_w; Q, F) > u_1(t_w; B, F)$ since $\phi_{1w} > 0$. This means that the *weak* type has no incentive to deviate in any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|B) = 1$ and hence Player 2 will choose NF after observing the deviation play B under region $\boldsymbol{A}_a$. Now we need to check whether the *strong* type is better off by deviation under this situation. Note
    $$u_1(t_s; Q, F) = -\phi_{1s} > u_1(t_s; B, NF) = \mu + X_1'\beta_1 - \varepsilon_1$$

    under $\boldsymbol{S}_4$. Thus, the *strong* type will not deviate. Therefore, the equilibrium outcome survives the Intuitive Criterion.

- Under region $\boldsymbol{A}_b = \{\varepsilon_1 < \mu + X_1'\beta_1 - \phi_{1s}$ and $X_2'\beta_2 - \phi_{2w} < \varepsilon_1 < X_2'\beta_2 + (\phi_{2s} + \phi_{2w})p - \phi_{2w}\}$, we show that the pooling with (Q, Q) cannot survive the Intuitive Criterion of Cho and Kreps (1987) while the pooling with (B, B) survives it.

    - Pooling with $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ with $A_{2|Q} = NF$

    Note $u_1(t_w; Q, NF) > u_1(t_w; B, F)$ since $\mu + X_1'\beta_1 - \varepsilon_1 + \phi_{1w} > 0$ under $\boldsymbol{A}_b$ and note $u_1(t_w; Q, NF) > u_1(t_w; B, NF)$ since $\phi_{1w} > 0$. This means that the *weak* type has no incentive to deviate in any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|B) = 1$ and hence Player 2 will choose NF after observing the deviation play B under region $\boldsymbol{A}_b$. Now we need to check whether the *strong* type is better off by deviation under this situation. Note

    $$u_1(t_s; Q, NF) = \mu + X_1'\beta_1 - \varepsilon_1 - \phi_{1s} < u_1(t_s; B, NF) = \mu + X_1'\beta_1 - \varepsilon_1$$

    since $\phi_{1s} > 0$. Thus, the *strong* type will deviate for sure. Therefore, the equilibrium outcome fails the Intuitive Criterion.

    - Pooling with $(A_{1t_s}, A_{1t_w}) = (B, B)$ with $A_{2|B} = NF$

    Note $u_1(t_s; B, NF) > u_1(t_s; Q, NF)$ since $\phi_{1s} > 0$ and note $u_1(t_s; B, NF) > u_1(t_s; Q, F)$ under $\boldsymbol{A}_b$. This means that the *strong* type has no incentive to deviate in any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|Q) = 0$ and hence Player 2 will choose F after observing the deviation play Q under region $\boldsymbol{A}_b$. Now we need to check whether the *weak* type is better off by deviation under this situation. Note

    $$u_1(t_w; B, NF) = \mu + X_1'\beta_1 - \varepsilon_1 - \phi_{1w} > u_1(t_w; Q, F) = 0$$

    under region $\boldsymbol{S}_5$. Thus, the *weak* type will not deviate. Therefore, the equilibrium outcome survives the Intuitive Criterion.

- The separating equilibrium (B, Q) with $(Y_{1|B} = 1, Y_{1|Q} = 0)$ under region $\boldsymbol{S}_3$ cannot fail the Intuitive Criterion since none of Player 1 wants to deviate regardless of Player 2's action.

B   Game with Asymmetric Payoffs

For the game presented in Figure 2, we note that the payoffs structure is restrictive. We relax this restriction as in the game of Figure A1 by allowing μ and β_1 to be different for different types. It turns out that we can achieve uniqueness of equilibrium even under asymmetric payoffs by imposing Conditions 1-2 (see the Appendix D.3).

Condition 1 $\mu_w + X_1'\beta_w - \phi_{1w} < \mu_s + X_1'\beta_s + \phi_{1s}$ *for all* $X_1 \in \mathbb{S}(X_1)$.

Condition 2 $\mu_s + X_1'\beta_s - \phi_{1s} < \mu_w + X_1'\beta_w + \phi_{1w}$ *for all* $X_1 \in \mathbb{S}(X_1)$.

These conditions hold immediately in the symmetric payoffs case ($\mu_w = \mu_s, \beta_s = \beta_w$) since $\phi_{1s}, \phi_{1w} > 0$. We summarize the result as

Corollary B.1 *Suppose Conditions 1-2 hold. Further suppose Assumptions **IS** and **SA-1** hold and that $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). Suppose each Player plays only one equilibrium that survives the refinement of Cho and Kreps (1987), when there exist multiple equilibria. Then, there exist unique equilibrium for each region of $(\varepsilon_1, \varepsilon_2)$ given X.*

See the Appendix D.3 for the proof. Figure A2 illustrates uniqueness of equilibrium for the game with **IS** and asymmetric payoffs. We note that all the estimation strategies considered in this paper are still valid for the asymmetric payoffs case as long as Assumption **SA-Asym** holds. However, even if Assumption **SA-Asym** is violated for certain observations in the data, we can still estimate the game model using a trimming device that trims out those observations violating Assumption **SA-Asym** under certain conditions.

Assumption B.1 (*SA-Asym*)
$\mu_w + X_1'\beta_w - \phi_{1w} < \mu_s + X_1'\beta_s + \phi_{1s}$ *and* $\mu_s + X_1'\beta_s - \phi_{1s} < \mu_w + X_1'\beta_w + \phi_{1w}$ *hold for all* $X_1 \in \mathbb{S}(X_1)$ *and for all* $(\mu_s, \mu_w, \beta_s, \beta_w, \phi_{1s}, \phi_{1w})$ *in the parameter space.*

We note that this assumption is embedded when we derive the appropriate conditional probabilities in the following section. Again note that Assumption **SA-Asym** holds immediately in the symmetric payoffs game ($\mu_w = \mu_s, \beta_s = \beta_w$) since $\phi_{1s}, \phi_{1w} > 0$.
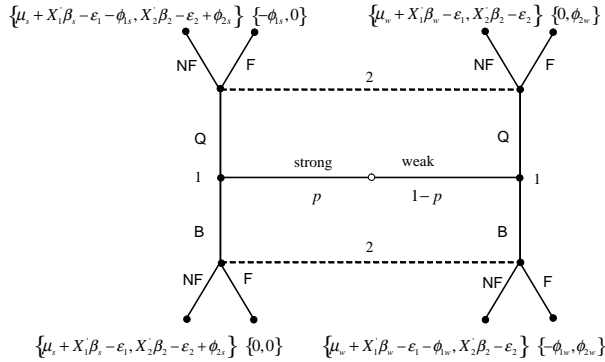
FIGURE A1. Structure of the Game with Asymmetric Payoffs
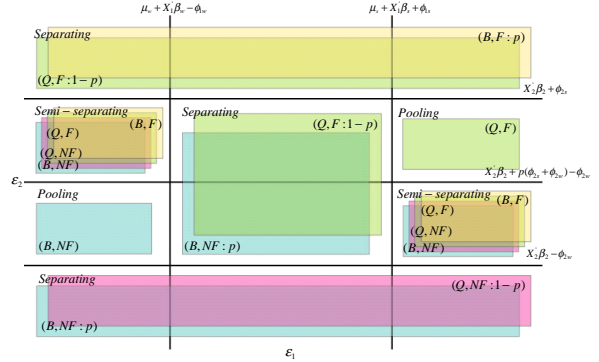
FIGURE A2. Uniqueness of Equilibrium under Asymmetric Payoffs

C   Conditional Probabilities of Four Observed Outcomes

Summarizing the result derived in the Appendix D.2, here we provide the conditional probabilities of four observed outcomes in the game with **IS-A** and **SA-2** under asymmetric payoffs of Figure A1. For the game with **IS** and **SA-1**, we obtain the same conditional probabilities in replace of $p(Z)$ with p and W with X, respectively. Also note that the corresponding conditional probabilities of the symmetric payoffs game (Figure 2) are easily obtained by replacing β_s & β_w with β_1 and μ_s & μ_w with μ, respectively.

1 $(Y_1 = 1, Y_2 = 1) : (B, NF)$

It happens under $\boldsymbol{S}_5$ with probability one (pooling), under $\boldsymbol{S}_3 \cup \boldsymbol{S}_7$ with probability $p(Z)$ (separating), and under $\boldsymbol{S}_2 \cup \boldsymbol{S}_6$ (semi-separating). From these, we have

$$\Pr\left(Y_1 = 1, Y_2 = 1 | W, \alpha\right) =$$
$$G_1\left(\mu_w + X_1'\beta_w - \phi_{1w}\right)\left(G_2\left(X_2'\beta_2 + p(Z)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right) - G_2\left(X_2'\beta_2 - \phi_{2w}\right)\right)$$
$$+p(Z)\left(G_1\left(\mu_s + X_1'\beta_s + \phi_{1s}\right) - G_1\left(\mu_w + X_1'\beta_w - \phi_{1w}\right)\right)\left(G_2\left(X_2'\beta_2 + \phi_{2s}\right) - G_2\left(X_2'\beta_2 - \phi_{2w}\right)\right)$$
$$+p(Z)G_2\left(X_2'\beta_2 - \phi_{2w}\right)$$
$$+ \begin{array}{l} \int_0^1 \left(p(Z) + (1-p(Z))\mu_B^w\right) g_2\left(X_2'\beta_2 + \left(\frac{p(Z)}{p(Z) + \mu_B^w \cdot (1-p(Z))}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(p(Z)+(1-p(Z))\mu_B^w\right)^2}d\mu_B^w \\ \times \int_0^1 \widetilde{\sigma}_2 g_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}d\widetilde{\sigma}_2 \end{array}$$
$$+ \begin{array}{l} \int_0^1 p(Z)\mu_B^s g_2\left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p(Z)}{(1-\mu_B^s)p(Z)+(1-p(Z))}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(1-\mu_B^s p(Z)\right)^2}d\mu_B^s \\ \times \int_0^1 \sigma_2 g_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{1-\sigma_2}\right)\frac{\phi_{1s}}{(1-\sigma_2)^2}d\sigma_2 \end{array}$$

2 $(Y_1 = 1, Y_2 = 0) : (B, F)$

It happens under $\boldsymbol{S}_1$ with probability $p(Z)$ (separating) and under $\boldsymbol{S}_2 \cup \boldsymbol{S}_6$ (semi-separating). From these, we obtain

$$\Pr\left(Y_1 = 1, Y_2 = 0 | W, \alpha\right) =$$
$$p(Z)\left(1 - G_2\left(X_2'\beta_2 + \phi_{2s}\right)\right)$$
$$+ \begin{array}{l} \int_0^1 \left(p(Z) + (1-p(Z))\mu_B^w\right) g_2\left(X_2'\beta_2 + \left(\frac{p(Z)}{p(Z) + \mu_B^w \cdot (1-p(Z))}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(p(Z)+(1-p(Z))\mu_B^w\right)^2}d\mu_B^w \\ \times \int_0^1 \left(1 - \widetilde{\sigma}_2\right) g_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}d\widetilde{\sigma}_2 \end{array}$$
$$+ \begin{array}{l} \int_0^1 p(Z)\mu_B^s g_2\left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p(Z)}{(1-\mu_B^s)p(Z)+(1-p(Z))}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(1-\mu_B^s p(Z)\right)^2}d\mu_B^s \\ \times \int_0^1 \left(1 - \sigma_2\right) g_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{1-\sigma_2}\right)\frac{\phi_{1s}}{(1-\sigma_2)^2}d\sigma_2 \end{array}$$

3 $(Y_1 = 0, Y_2 = 1) : (Q, NF)$

It happens under $\boldsymbol{S}_7$ with probability $1 - p(Z)$ (separating), and under $\boldsymbol{S}_2 \cup \boldsymbol{S}_6$ (semi-separating). From these, we have

$$\Pr\left(Y_1 = 0, Y_2 = 1 | W, \alpha\right) =$$
$$\left(1 - p(Z)\right) G_2\left(X_2'\beta_2 - \phi_{2w}\right)$$
$$+ \begin{array}{l} \int_0^1 \left(1-p(Z)\right)\left(1 - \mu_B^w\right) g_2\left(X_2'\beta_2 + \left(\frac{p(Z)}{p(Z) + \mu_B^w \cdot (1-p(Z))}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(p(Z)+\mu_B^w \cdot (1-p(Z))\right)^2}d\mu_B^w \\ \times \int_0^1 \widetilde{\sigma}_2 g_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}d\widetilde{\sigma}_2 \end{array}$$
$$+ \begin{array}{l} \int_0^1 \left(1 - \mu_B^s p(Z)\right) g_2\left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p(Z)}{1-\mu_B^s p(Z)}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(1-\mu_B^s p(Z)\right)^2}d\mu_B^s \\ \times \int_0^1 \sigma_2 g_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{1-\sigma_2}\right)\frac{\phi_{1s}}{(1-\sigma_2)^2}d\sigma_2 \end{array}$$

4 $(Y_1 = 0, Y_2 = 0) : (Q, F)$

It happens under $\boldsymbol{S}_4$ with probability one (pooling), under $\boldsymbol{S}_1 \cup \boldsymbol{S}_3$ with probability $1 - p(Z)$ (separating), and under $\boldsymbol{S}_2 \cup \boldsymbol{S}_6$ (semi-separating). From these, we conclude

$$\Pr\left(Y_1 = 0, Y_2 = 0 | W, \alpha\right) =$$
$$\left(1 - G_1\left(\mu_s + X_1'\beta_s + \phi_{1s}\right)\right)\left(G_2\left(X_2'\beta_2 + \phi_{2s}\right) - G_2\left(X_2'\beta_2 + p(Z)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\right)$$
$$+\left(1 - p(Z)\right)\left(1 - G_2\left(X_2'\beta_2 + \phi_{2s}\right)\right)$$
$$+\left(1 - p(Z)\right)\left(G_1\left(\mu_s + X_1'\beta_s + \phi_{1s}\right) - G_1\left(\mu_w + X_1'\beta_w - \phi_{1w}\right)\right)\left(G_2\left(X_2'\beta_2 + \phi_{2s}\right) - G_2\left(X_2'\beta_2 - \phi_{2w}\right)\right)$$
$$+ \begin{array}{l} \int_0^1 \left(1-p(Z)\right)\left(1 - \mu_B^w\right) g_2\left(X_2'\beta_2 + \left(\frac{p(Z)}{p(Z) + \mu_B^w \cdot (1-p(Z))}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(p(Z)+\mu_B^w \cdot (1-p(Z))\right)^2}d\mu_B^w \\ \times \int_0^1 \left(1 - \widetilde{\sigma}_2\right) g_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}d\widetilde{\sigma}_2 \end{array}$$
$$+ \begin{array}{l} \int_0^1 \left(1 - \mu_B^s p(Z)\right) g_2\left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p(Z)}{1-\mu_B^s p(Z)}\right)\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}\right)\frac{p(Z)(1-p(Z))(\phi_{2s}+\phi_{2w})}{\left(1-\mu_B^s p(Z)\right)^2}d\mu_B^s \\ \times \int_0^1 \left(1 - \sigma_2\right) g_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{1-\sigma_2}\right)\frac{\phi_{1s}}{(1-\sigma_2)^2}d\sigma_2 \end{array}$$
.

Appendix II

D   Perfect Bayesian Equilibrium

D.1   Definition of PBE

We let $P(h)$ and $A(h)$ be the player function and the action set at the history h, respectively.

Definition D.1 *(Osborne and Rubinstein (p.233, 1994))*
Let $\langle \Gamma, (\Xi_i), (p_i), (u_i) \rangle$ be a Bayesian extensive game with observable actions, where $\Gamma = \langle N, H, P \rangle$. A pair $((\sigma_i), (\mu_i)) = \left((\sigma_i(t_i))_{i \in N, t_i \in \Xi_i}, (\mu_i(h))_{i \in N, h \in H \backslash \boldsymbol{T}} \right)$, where $\sigma_i(t_i)$ is a behavioral strategy of player i in Γ, $\mu_i(h)$ is a probability measure on Ξ_i, and $\boldsymbol{T}$ is the terminal history, is a perfect Bayesian equilibrium of the game if the following conditions are satisfied

- *Sequential Rationality: For every nonterminal history $h \in H \backslash \boldsymbol{T}$ every player $i \in P(h)$, every $t_i \in \Xi_i$, $u_i(\sigma_{-i}, \sigma_i(t_i), \mu_{-i}|h)$ is at least as good for type t_i as $u_i(\sigma_{-i}, s_i, \mu_{-i}|h)$ for any strategy s_i of player i in Γ.*
- *Correct initial beliefs: $\mu_i(\varnothing) = p_i$ for each $i \in N$*
- *Action-determined beliefs: If $i \notin P(h)$ and $a \in A(h)$ then $\mu_i(h, a) = \mu_i(h)$; if $i \in P(h)$, $a \in A(h)$, $a' \in A(h)$, and $a_i = a'_i$ then $\mu_i(h, a) = \mu_i(h, a')$.*
- *Bayesian updating: If $i \in P(h)$ and a_i is in the support of $\sigma_i(t_i)(h)$ for some t_i in the support of $t_i(h)$ then for any $t'_i \in \Xi_i$ we have*
$$\mu_i(h, a)(t'_i) = \frac{\sigma_i(t'_i)(h)(a_i) \cdot \mu_i(h)(t'_i)}{\sum_{t_i \in \Xi_i} \sigma_i(t_i)(h)(a_i) \cdot \mu_i(h)(t_i)}.$$

D.2   PBE of the Game G with IS

We derive equilibria of the game with asymmetric payoffs as in Figure A1. Equilibria of the game with symmetric payoffs ($\beta_s = \beta_w$, $\mu_s = \mu_w$) are easily obtained from the results in this section. We impose $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). We let $E_i[u_i(t_1; A_1, A_2)]$ be the expected payoffs of Player i based on Player i's information for $i \in \{1, 2\}$.

D.2.1   Pooling Equilibria

P-1) Pooling Equilibrium (both t_s and t_w choose B):
   Then, Player 2 does not update its belief and thus the posterior equals to the prior belief: $\mu_2(t_1 = t_s|B) = p$ and $\mu_2(t_1 = t_w|B) = 1 - p$. Thus, the expected payoffs of Player 2 from choosing each action on the equilibrium path will be
$$E_2[u_2(t_1; B, NF)] = X'_2 \beta_2 + p \cdot \phi_{2s} - \varepsilon_2 \quad \text{and} \quad E_2[u_2(t_1; B, F)] = (1 - p) \cdot \phi_{2w}$$

and hence $Y_{2|B} = \mathbf{1}\{X'_2 \beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} - \varepsilon_2 \geq 0\}$ on the equilibrium path. The resulting payoffs of each type of Player 1 on the equilibrium path will be

	$Y_{2|B} = 1$	$Y_{2|B} = 0$
$u_1(t_s; B, \cdot)$	$\mu_s + X'_1 \beta_s - \varepsilon_1$	0
$u_1(t_w; B, \cdot)$	$\mu_w + X'_1 \beta_w - \varepsilon_1 - \phi_{1w}$	$-\phi_{1w}$

1) To have $Y_{2|B} = 1$ as an equilibrium with pooling $(A_{1t_s}, A_{1t_w}) = (B, B)$:

P1-11: t_s should have no incentive to deviate. If $u_1(t_s; B, NF) = \mu_s + X'_1 \beta_s - \varepsilon_1 \geq u_1(t_s, Q, F) = -\phi_{1s}$, then no matter what the value of $\mu_2(t_1 = t_s|Q)$, t_s has no incentive to deviate by construction. Now assume $\mu_s + X'_1 \beta_s - \varepsilon_1 \leq -\phi_{1s}$. Player 1 expect that Player 2 with $\overline{\mu}_2 \equiv \mu_2(t_1 = t_s|Q)$ as off-the-equilibrium belief will choose $Y_{2|Q} = \mathbf{1}\{X'_2 \beta_2 + \overline{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w} - \varepsilon_2 \geq 0\}$. To make Player 1 not deviate, the belief should induce Player 2 chooses NF after observing Q. It is required that $Y_{2|Q} = 1$ while $Y_{2|B} = 1$. This holds when $\min(X'_2 \beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}, X'_2 \beta_2 + \overline{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w}) \geq \varepsilon_2$.

P1-12: t_w should have no incentive to deviate. Since $u_1(t_w; B, NF) < u_1(t_w; Q, NF)$, Player 1 of type t_w has an incentive to deviate unless $u_1(t_w; B, NF) \geq u_1(t_w; Q, F)$ and Player 2 chooses F after seeing Q (while choosing NF if seeing B). Not-to-deviate conditions require that $\mu_w + X_1'\beta_w - \varepsilon_1 - \phi_{1w} \geq 0$ and $X_2'\beta_2 + \overline{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}$.

Combining P1-11 and P1-12, we conclude that $Y_{2|B} = 1$ and $Y_{2|Q} = 0$ with pooling $(A_{1t_s}, A_{1t_w}) = (B, B)$ can be supported as an equilibrium where Player 2 assigns a belief with $\overline{\mu}_2 \equiv \mu_2(t_1 = t_s|Q)$ under

$$\varepsilon_1 \leq \min\left(\mu_w + X_1'\beta_w - \phi_{1w}, \mu_s + X_1'\beta_s + \phi_{1s}\right) \text{ and}$$
$$X_2'\beta_2 + \overline{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}, \tag{34}$$

which requires $\overline{\mu}_2 < p$[33] for the existence of this equilibrium. Note that for any $\overline{\mu}_2 < p$ satisfying (34), the equilibrium is supported[34] and hence regions of $(\varepsilon_1, \varepsilon_2)$ that support the equilibrium are actually

$$\varepsilon_1 \leq \min\left(\mu_w + X_1'\beta_w - \phi_{1w}, \mu_s + X_1'\beta_s + \phi_{1s}\right) \text{ and } X_2'\beta_2 - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}.$$

2) To have $Y_{2|B} = 0$ as an equilibrium with pooling $(A_{1t_s}, A_{1t_w}) = (B, B)$:

We first assume that Player 1 expects that Player 2 will have $\overline{\overline{\mu}}_2 \equiv \mu_2(t_1 = t_s|Q)$ as off-the-equilibrium belief after observing Q.

P1-21: t_s should have no incentive to deviate. Player 1 of type t_s will have no incentive to deviate from the equilibrium in either of two cases
  i. $u_1(t_s; Q, NF) \leq u_1(t_s; B, F)$. This holds when $\varepsilon_1 \geq \mu_s + X_1'\beta_s - \phi_{1s}$.
  ii. $u_1(t_s; Q, NF) \geq u_1(t_s; B, F)$ but $Y_{2|Q} = 0$. This holds when $\varepsilon_1 \leq \mu_s + X_1'\beta_s - \phi_{1s}$ and $X_2'\beta_2 + \overline{\overline{\mu}}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w} \leq \varepsilon_2$.
P1-22: t_w should have no incentive to deviate. Since $u_1(t_w; B, F) < u_1(t_w; Q, F)$, Player 1 of type t_w has an incentive to deviate unless $u_1(t_w; Q, NF) \leq u_1(t_w; B, F)$ and $Y_{2|Q} = 1$. This requires $\varepsilon_1 \geq \mu_w + X_1'\beta_w + \phi_{1w}$ and $X_2'\beta_2 + \overline{\overline{\mu}}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w} \geq \varepsilon_2$ for not-to-deviate play.

Combining P1-21 and P1-22, we conclude that under

$$\varepsilon_1 \geq \max\left(\mu_s + X_1'\beta_s - \phi_{1s}, \varepsilon_1, \mu_w + X_1'\beta_w + \phi_{1w}\right) \text{ and}$$
$$X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + \overline{\overline{\mu}}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w}, \tag{35}$$

the equilibrium of $Y_{2|B} = 0$ and $Y_{2|Q} = 1$ with pooling $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ can be supported as an equilibrium where Player 2 holds off-the-equilibrium belief $\overline{\overline{\mu}}_2 \equiv \mu_2(t_1 = t_s|Q)$. Note that for any $\overline{\overline{\mu}}_2 > p$ that satisfies (35), the equilibrium is supported[35] and hence regions of $(\varepsilon_1, \varepsilon_2)$ that support the equilibrium are actually

$$\varepsilon_1 \geq \max\left(\mu_s + X_1'\beta_s - \phi_s, \mu_w + X_1'\beta_w + \phi_w\right) \text{ and } X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + \phi_{2s}.$$

P-2) Pooling Equilibrium (both t_s and t_w choose Q):

In this case, Player 2 does not update its belief and thus the posterior equals to the prior belief: $\mu_2(t_1 = t_s|Q) = p$ and $\mu_2(t_1 = t_w|Q) = 1 - p$. Thus, the expected payoffs of Player 2 from choosing each action on the equilibrium path will be

$$E_2[u_2(t_1; Q, NF)] = X_2'\beta_2 + p \cdot \phi_{2s} - \varepsilon_2 \text{ and } E_2[u_2(t_1; Q, F)] : (1-p) \cdot \phi_{2w}$$

and hence $Y_{2|Q} = \mathbf{1}\{X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} - \varepsilon_2 \geq 0\}$ on the equilibrium path. The resulting payoffs of each type of Player 1 on the equilibrium path will be

$$\begin{array}{ccc} & Y_{2|Q} = 1 & Y_{2|Q} = 0 \\ u_1(t_s; Q, \cdot) & \mu_s + X_1'\beta_s - \varepsilon_1 - \phi_{1s} & -\phi_{1s} \\ u_1(t_w; Q, \cdot) & \mu_w + X_1'\beta_w - \varepsilon_1 & 0 \end{array}.$$

1) To have $Y_{2|Q} = 1$ as an equilibrium with pooling $(t_s, t_w) = (Q, Q)$:

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[33] Note that $\phi_{2s} + \phi_{2w} > 0$.

[34] This means that for any realization of X_2 and ε_2 satisfying $\varepsilon_2 \leq X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}$, there exist values of $\overline{\mu}_2 = \overline{\mu}_2(X_2'\beta_2, p, \phi_{2s}, \phi_{2w}, \varepsilon_2)$, $0 \leq \overline{\mu}_2 < p$ such that (34) holds.

[35] Similarly as before, this means that for any realization of X_2 and ε_2 satisfying $\varepsilon_2 \geq X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}$, there exist values of $\overline{\overline{\mu}}_2 = \overline{\overline{\mu}}_2(X_2'\beta_2, p, \phi_{2s}, \phi_{2w}, \varepsilon_2)$, $p < \overline{\overline{\mu}}_2 \leq 1$ such that (35) holds.

P2-11: t_w should have no incentive to deviate. If $u_1(t_w; Q, NF) = \mu_w + X_1'\beta_w - \varepsilon_1 \geq u_1(t_w; B, F) = -\phi_{1w}$, then no matter what the value of $\mu_2(t_1 = t_s|B)$, t_w has no incentive to deviate by construction. Now assume $\mu_w + X_1'\beta_w - \varepsilon_1 \leq -\phi_{1w}$. Player 2 with $\widetilde{\mu}_2 \equiv \mu_2(t_1 = t_s|B)$ as off-the-equilibrium belief will choose $Y_{2|B} = \mathbf{1}\left\{X_2'\beta_2 + \widetilde{\mu}_2\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w} - \varepsilon_2 \geq 0\right\}$. Thus, Player 1 of type t_w will have no incentive to deviate from the equilibrium if Player 2 chooses NF after observing B. This requires that $Y_{2|B} = 1$ together with $Y_{2|Q} = 1$:

$$\varepsilon_2 \leq \min(X_2'\beta_2 + p\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}, X_2'\beta_2 + \widetilde{\mu}_2\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}).$$

P2-12: t_s should have no incentive to deviate. Since $u_1(t_s; Q, NF) < u_1(t_s; B, NF)$, Player 1 of type t_s has an incentive to deviate unless $u_1(t_s; Q, NF) \geq u_1(t_s; B, F)$ and Player 2 chooses F after seeing B (while choosing NF if seeing Q). Not-to-deviate conditions require that $\mu_s + X_1'\beta_1 - \varepsilon_1 - \phi_{1s} \geq 0$ and $X_2'\beta_2 + \widetilde{\mu}_2\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + p\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}$.

Combining P2-11 and P2-12, we conclude that $Y_{2|Q} = 1$ and $Y_{2|B} = 0$ with pooling $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ can be supported as an equilibrium where Player 2 holds off-the-equilibrium belief $\widetilde{\mu}_2 \equiv \mu_2(t_1 = t_s|B)$ under

$$\varepsilon_1 \leq \min\left(\mu_s + X_1'\beta_s - \phi_{1s}, \mu_w + X_1'\beta_w + \phi_{1w}\right) \text{ and}$$
$$X_2'\beta_2 + \widetilde{\mu}_2\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + p\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}, \tag{36}$$

which requires $\widetilde{\mu}_2 < p$ for the existence of this equilibrium. Note that for any $\widetilde{\mu}_2 < p$ that satisfies (36), the equilibrium is supported similarly as before and hence regions of $(\varepsilon_1, \varepsilon_2)$ that support the equilibrium are actually

$$\varepsilon_1 \leq \min\left(\mu_s + X_1'\beta_s - \phi_{1s}, \mu_w + X_1'\beta_w + \phi_{1w}\right) \text{ and } X_2'\beta_2 - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + p\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}.$$

2) To have $Y_{2|Q} = 0$ as an equilibrium with pooling $(A_{1t_s}, A_{1t_w}) = (Q, Q)$:

We first assume that Player 1 expect that Player 2 will have $\widetilde{\widetilde{\mu}}_2 \equiv \mu_2(t_1 = t_s|B)$ as off-the-equilibrium belief after observing B.

P2-21: t_w should have no incentive to deviate. Player 1 of type t_w will have no incentive to deviate from the equilibrium in either of two cases
  i. $u_1(t_w; B, NF) \leq u_1(t_w; Q, F)$. This holds when $\varepsilon_1 \geq \mu_w + X_1'\beta_w - \phi_{1w}$.
  ii. $u_1(t_w; Q, NF) \geq u_1(t_w; Q, F)$ but $Y_{2|B} = 0$.
P2-22: t_s should have no incentive to deviate. Since $u_1(t_s, Q, F) < u_1(t_s, B, F)$, Player 1 of type t_s has an incentive to deviate unless $u_1(t_s, B, NF) \leq u_1(t_s, Q, F)$ and $Y_{2|B} = 1$.

Combining P2-21 and P2-22, we conclude that $Y_{2|Q} = 0$ and $Y_{2|B} = 1$ with pooling $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ can be supported as an equilibrium under

$$\varepsilon_1 \geq \max\left(\mu_w + X_1'\beta_w - \phi_{1w}, \mu_s + X_1'\beta_s + \phi_{1s}\right) \text{ and}$$
$$X_2'\beta_2 + p\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + \widetilde{\widetilde{\mu}}_2\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w}. \tag{37}$$

Similarly to the previous case, note that for any $\widetilde{\widetilde{\mu}}_2 > p$ satisfying (37), the equilibrium is supported and hence regions of $(\varepsilon_1, \varepsilon_2)$ that support the equilibrium are actually

$$\varepsilon_1 \geq \max\left(\mu_w + X_1'\beta_w - \phi_{1w}, \mu_s + X_1'\beta_s + \phi_{1s}\right) \text{ and } X_2'\beta_2 + p\left(\phi_{2s} + \phi_{2w}\right) - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + \phi_{2s}.$$

D.2.2   Separating Equilibria

S-1) Separating Equilibrium (t_s chooses B and t_w choose Q):
In the separating equilibrium, Player 2 has complete information once it observes the signal. Thus, $\mu_2(t_1 = t_s|B) = 1$ and $\mu_2(t_1 = t_s|Q) = 0$. Upon seeing B Player 2 will choose NF if $u_2(t_s; B, NF) \geq u_2(t_s; B, F)$ and choose F otherwise. This implies $Y_{2|B} \equiv \mathbf{1}\left\{X_2'\beta_2 + \phi_{2s} - \varepsilon_2 \geq 0\right\}$. Similarly upon seeing Q Player 2 will have $Y_{2|Q} = \mathbf{1}\left\{X_2'\beta_2 - \varepsilon_2 \geq \phi_{2w}\right\}$.
  1) To have $\left(Y_{2|B} = 1, Y_{2|Q} = 1\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (B, Q)$: (When $\varepsilon_2 \leq X_2'\beta_2 - \phi_{2w}$)

S1-11: t_s should have no incentive to deviate. It requires $u_1(t_s; B, NF) = \mu_s + X_1'\beta_s - \varepsilon_1 \geq u_1(t_s; Q, NF) = \mu_s + X_1'\beta_s - \varepsilon_1 - \phi_{1s}$, which holds for all ε_1.

S1-12: t_w should have no incentive to deviate. It requires $u_1(t_w; Q, NF) = \mu_w + X_1'\beta_w - \varepsilon_1 \geq u_1(t_w; B, NF) = \mu_w + X_1'\beta_w - \varepsilon_1 - \phi_{1w}$, which also holds for all ε_1.

Combining S1-11 and S1-12, we conclude that $\left(Y_{2|B} = 1, Y_{2|Q} = 1\right)$ with separating $(A_{1t_s}, A_{1t_w}) = (B, Q)$ can be supported as an equilibrium as long as $\varepsilon_2 \leq X_2'\beta_2 - \phi_{2w}$.

2) To have $\left(Y_{2|B} = 1, Y_{2|Q} = 0\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (B, Q)$: (When $X_2'\beta_2 - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + \phi_{2s}$)

S1-21: t_s should have no incentive to deviate. It requires $u_1(t_s; B, NF) = \mu_s + X_1'\beta_s - \varepsilon_1 \geq u_1(t_s; Q, F) = -\phi_{1s}$, which holds for $\varepsilon_1 \leq \mu_s + X_1'\beta_s + \phi_{1s}$.

S1-22: t_w should have no incentive to deviate. It requires $u_1(t_w; Q, F) = 0 \geq u_1(t_w; B, NF) = \mu_w + X_1'\beta_w - \varepsilon_1 - \phi_{1w}$, which holds for $\varepsilon_1 \geq \mu_w + X_1'\beta_w - \phi_{1w}$.

Combining S1-21 and S1-22, we conclude that $\left(Y_{2|B} = 1, Y_{2|Q} = 0\right)$ with separating $(A_{1t_s}, A_{1t_w}) = (B, Q)$ can be supported as an equilibrium under[36]

$$X_2'\beta_2 - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + \phi_{2s} \text{ and } \mu_w + X_1'\beta_w - \phi_{1w} \leq \varepsilon_1 \leq \mu_s + X_1'\beta_s + \phi_{1s}.$$

3) To have $\left(Y_{2|B} = 0, Y_{2|Q} = 1\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (B, Q)$: There is no such ε_2 that supports this equilibrium since $\phi_{2s} \geq 0$, $\phi_{2w} \geq 0$, and $\phi_{2s} \cdot \phi_{2w} \neq 0$.

4) To have $\left(Y_{2|B} = 0, Y_{2|Q} = 0\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (B, Q)$: (When $\varepsilon_2 \geq X_2'\beta_2 + \phi_{2s}$)

S1-41: t_s should have no incentive to deviate. It requires $u_1(t_s; B, F) = 0 \geq u_1(t_s; Q, F) = -\phi_{1s}$, which holds for all ε_1.

S1-42: t_w should have no incentive to deviate. It requires $u_1(t_w; Q, F) = 0 \geq u_1(t_w; B, F) = -\phi_{1w}$, which also holds for all ε_1.

Combining S1-41 and S1-42, we conclude that $\left(Y_{2|B} = 0, Y_{2|Q} = 0\right)$ with separating $(A_{1t_s}, A_{1t_w}) = (B, Q)$ can be supported as an equilibrium as long as $\varepsilon_2 \geq X_2'\beta_2 + \phi_{2s}$.

S-2) Separating Equilibrium (t_s chooses Q and t_w choose B):
In the separating equilibrium, Player 2 has complete information once it observes the signal. Thus, $\mu_2(t_1 = t_s|B) = 0$ and $\mu_2(t_1 = t_s|Q) = 1$. Upon seeing B Player 2 will choose NF if $u_2(t_w, B, NF) \geq u_2(t_w, B, F)$ and choose F otherwise. This implies $Y_{2|B} \equiv \mathbf{1}\left\{X_2'\beta_2 - \varepsilon_2 \geq \phi_{2w}|B\right\}$. Similarly upon seeing Q Player 2 will have $Y_{2|Q} = \mathbf{1}\left\{X_2'\beta_2 - \varepsilon_2 + \phi_{2s} \geq 0\right\}$.

1) To have $\left(Y_{2|B} = 1, Y_{2|Q} = 1\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (Q, B)$: (When $\varepsilon_2 \leq X_2'\beta_2 - \phi_{2w}$)
t_s should have no incentive to deviate. It requires $u_1(t_s; Q, NF) = \mu_s + X_1'\beta_s - \varepsilon_1 - \phi_{1s} \geq u_1(t_s; B, NF) = \mu_s + X_1'\beta_s - \varepsilon_1$. Hence there is no such ε_1 that satisfies this condition since $\phi_{1s} > 0$. Thus, $Y_{2|B} = 1, Y_{2|Q} = 1$ with separating $(A_{1t_s}, A_{1t_w}) = (Q, B)$ cannot be an equilibrium.

2) To have $\left(Y_{2|B} = 1, Y_{2|Q} = 0\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (Q, B)$: There is no such ε_2 that supports this equilibrium since $\phi_{2s} \geq 0$, $\phi_{2w} \geq 0$, and $\phi_{2s} \cdot \phi_{2w} \neq 0$.

3) To have $\left(Y_{2|B} = 0, Y_{2|Q} = 1\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (Q, B)$: (When $X_2'\beta_2 - \phi_{2w} \leq \varepsilon_2 \leq X_2'\beta_2 + \phi_{2s}$)

S2-21: t_s should have no incentive to deviate. It requires $u_1(t_s; Q, NF) = \mu_s + X_1'\beta_s - \varepsilon_1 - \phi_{1s} \geq u_1(t_s; B, F) = 0$, which holds for $\varepsilon_1 \leq \mu_s + X_1'\beta_s - \phi_{1s}$.

S2-22: t_w should have no incentive to deviate. It requires $u_1(t_w; B, F) = -\phi_{1w} \geq u_1(t_w; Q, NF) = \mu_w + X_1'\beta_w - \varepsilon_1$, which holds for $\varepsilon_1 \geq \mu_w + X_1'\beta_w + \phi_{1w}$.

Combining S1-21 and S1-22, we conclude that there is no such ε_1 satisfying both S1-21 and S-22 if $\mu_s + X_1'\beta_s - \phi_{1s} < \mu_w + X_1'\beta_w + \phi_{1w}$.

4) To have $\left(Y_{2|B} = 0, Y_{2|Q} = 0\right)$ as an equilibrium with separating $(A_{1t_s}, A_{1t_w}) = (Q, B)$: (When $\varepsilon_2 \geq X_2'\beta_2 + \phi_{2s}$)
t_s should have no incentive to deviate. It requires $u_1(t_s; Q, F) = -\phi_{1s} \geq u_1(t_s; B, F) = 0$. Hence there is no such ε_1 that satisfies this condition since $\phi_{1s} > 0$. Thus, $Y_{2|B} = 0, Y_{2|Q} = 0$ with separating $(A_{1t_s}, A_{1t_w}) = (Q, B)$ cannot be an equilibrium.

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[36]If $\mu_w + X_1'\beta_w - \phi_w > \mu_s + X_1'\beta_s + \phi_s$, this equilibrium does not exist.

D.2.3 Semi-Separating Equilibria

We characterize the semi-separating equilibria in this section.

SS-1: **The weak type of Player 1 plays the separating equilibrium with Q and the strong type randomizes with $\sigma_{t_s}(B) = \mu_B^s$:**
Then,

$$\mu_2(t_1 = t_s | B) = \frac{\mu_B^s \cdot p}{\mu_B^s \cdot p} = 1 \text{ and } \mu_2^{ss} \equiv \mu_2(t_1 = t_s | Q) = \frac{(1 - \mu_B^s)\, p}{(1 - \mu_B^s)\, p + (1 - p)} \in (0, p)$$

which implies that the expected payoff of Player 2 for each action will be

$$
\begin{aligned}
E_2\left[u_2(t_1; Q, NF)\right] &= X_2'\beta_2 + \mu_2^{ss}\phi_{2s} - \varepsilon_2, \ E_2[u_2(t_1; Q, F)] = (1 - \mu_2^{ss})\,\phi_{2w}, \\
E_2[u_2(t_1; B, NF)] &= X_2'\beta_2 + \phi_{2s} - \varepsilon_2, \text{ and } E_2[u_2(t_1; B, F)] = 0.
\end{aligned}
$$

Player 2 will be indifferent between NF and F after observing Q if $\varepsilon_2 = X_2'\beta_2 + \mu_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}$ and under this, Player 2 will choose NF after seeing B since $\varepsilon_2 = X_2'\beta_2 + \mu_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}$ and $\mu_2^{ss} \in (0, p)$ implies that $E_2[u_2(t_1; B, NF)] > E_2[u_2(t_1; B, F)]$. Therefore, for the strong type of Player 1 to be indifferent between choosing B and Q, we require that

$$
\begin{aligned}
\mu_s + X_1'\beta_s - \varepsilon_1 &= u_1(t_s; B, NF) = E_1\left[u_1(t_s; Q, \cdot)\right] \\
&= \sigma_2(NF)\left(\mu_s + X_1'\beta_s - \varepsilon_1 - \phi_{1s}\right) + (1 - \sigma_2(NF))(-\phi_{1s})
\end{aligned}
$$

where $\sigma_2(NF)$ is the probability that Player 2 chooses NF. It follows that we require $\varepsilon_1 = \mu_s + X_1\beta_s + \frac{\phi_{1s}}{1 - \sigma_2(NF)}$ where $0 \leq \sigma_2(NF) \leq 1$. Noting ε_1 varies from $\mu_s + X_1\beta_s + \phi_{1s}$ to infinity and ε_2 varies from $X_2'\beta_2 - \phi_{2w}$ to $X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}$ since $\mu_2^{ss} \in (0, p)$, this implies that under region $\boldsymbol{S_6}$, there exist the semi-separating of **SS-1.** Finally note that we have

$$\varepsilon_1 = \mu_s + X_1\beta_s + \frac{\phi_{1s}}{1 - \sigma_2(NF)} \text{ and } \varepsilon_2 = X_2'\beta_2 + \left(\frac{(1 - \mu_B^s)\, p}{(1 - \mu_B^s)\, p + (1 - p)}\right)(\phi_{2s} + \phi_{2w}) - \phi_{2w}. \tag{38}$$

SS-2: **The strong type of Player 1 plays the separating equilibrium with B and the weak type randomizes with $\sigma_{t_w}(B) = \mu_B^w$:**
Then,

$$\widetilde{\mu}_2^{ss} = \mu_2(t_1 = t_s | B) = \frac{p}{p + \mu_B^w \cdot (1 - p)} \in (p, 1) \text{ and } \mu_2(t_1 = t_s | Q) = \frac{0}{(1 - \mu_B^w) \cdot (1 - p)} = 0$$

which implies that the expected payoff of Player 2 for each action will be

$$
\begin{aligned}
E_2[u_2(t_1; Q, NF)] &= X_2'\beta_2 - \varepsilon_2, \ E_2[u_2(t_1; Q, F)] = \phi_{2w}, \\
E_2[u_2(t_1; B, NF)] &= X_2'\beta_2 + \widetilde{\mu}_2^{ss}\phi_{2s} - \varepsilon_2, \text{ and } E_2[u_2(t_1; B, F)] = (1 - \widetilde{\mu}_2^{ss})\,\phi_{2w}.
\end{aligned}
$$

Player 2 will be indifferent between NF and F after observing B if $\varepsilon_2 = X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}$ and under this, Player 2 will choose F after seeing Q since $\varepsilon_2 = X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}$ and $\widetilde{\mu}_2^{ss} \in (p, 1)$ implies that $E_2[u_2(t_1; Q, NF)] < E_2[u_2(t_1; Q, F)]$. Therefore, for the weak type of Player 1 to be indifferent between choosing B and Q, we require that

$$0 = u_1(t_w; Q, F) = E_1\left[u_1(t_w; B, \cdot)\right] = \widetilde{\sigma}_2(NF)\left(\mu_w + X_1'\beta_w - \varepsilon_1 - \phi_w\right) + (1 - \widetilde{\sigma}_2(NF))(-\phi_w)$$

where $\widetilde{\sigma}_2(NF)$ is the probability that Player 2 chooses NF. It follows that we require $\varepsilon_1 = \mu_w + X_1\beta_w - \frac{\phi_w}{\widetilde{\sigma}_2(NF)}$ where $0 \leq \widetilde{\sigma}_2(NF) \leq 1$. This implies that under region $\boldsymbol{S_2}$, there exist the semi-separating of **SS-2** noting ε_1 varies from $\mu_w + X_1\beta_w - \phi_w$ to negative infinity and ε_2 varies from $X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}$ to $X_2'\beta_2 + \phi_{2s}$ since $\widetilde{\mu}_2^{ss} \in (p, 1)$. Finally note

$$\varepsilon_1 = \mu_w + X_1\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2(NF)} \text{ and } \varepsilon_2 = X_2'\beta_2 + \left(\frac{p}{p + \mu_B^w \cdot (1 - p)}\right)(\phi_{2s} + \phi_{2w}) - \phi_{2w}. \tag{39}$$

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D.2.4 Existence of Well-defined Likelihood for Semi-Separating

We let $\sigma_2 \equiv \sigma_2(NF)$ and $\widetilde{\sigma}_2 \equiv \widetilde{\sigma}_2(NF)$. In the equilibrium of **SS-1** under the region of $\boldsymbol{S}_6$, four observed outcomes can arise with following probabilities, respectively, denoted by

$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{S}_6, \mu_B^s, \sigma_2) = \int_{\boldsymbol{S}_6} p \cdot \mu_B^s(\varepsilon_2) \cdot \sigma_2(\varepsilon_1) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2$$
$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{S}_6, \mu_B^s, \sigma_2) = \int_{\boldsymbol{S}_6} p \cdot \mu_B^s(\varepsilon_2) \cdot (1 - \sigma_2(\varepsilon_1)) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2$$
$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{S}_6, \mu_B^s, \sigma_2) = \int_{\boldsymbol{S}_6} (p(1 - \mu_B^s(\varepsilon_2)) + (1-p)) \cdot \sigma_2(\varepsilon_1) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2$$
$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{S}_6, \mu_B^s, \sigma_2) = \int_{\boldsymbol{S}6} (p(1 - \mu_B^s(\varepsilon_2)) + (1-p)) \cdot (1 - \sigma_2(\varepsilon_1)) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2.$$

For example, $\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{S}_6, \mu_B^s, \sigma_2)$ is obtained using the following facts. Conditional on $(\varepsilon_1, \varepsilon_2)$, (B, NF) is observed when the nature draw the strong type with the probability p, the strong type plays B with the probability $\mu_B^s(\varepsilon_2)$, and Player 2 plays NF with probability $\sigma_2(\varepsilon_1)$ after observing B. Thus, the conditional probability of observing (B, NF) conditional on $(\varepsilon_1, \varepsilon_2)$ is given by

$$p \cdot \mu_B^s(\varepsilon_2) \cdot \sigma_2(\varepsilon_1) \tag{40}$$

and we obtain the unconditional probability by taking expectation of (40) with respect to $(\varepsilon_1, \varepsilon_2) \in \boldsymbol{S}_6$. Similarly, $\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{S}_6, \mu_B^s, \sigma_2)$ is obtained from the followings. Conditional on $(\varepsilon_1, \varepsilon_2)$, (Q, NF) is observed in two cases: 1) the nature draw the strong type with the probability p, the strong type plays Q with the probability $1 - \mu_B^s(\varepsilon_2)$, and Player 2 plays NF with probability $\sigma_2(\varepsilon_1)$ after observing Q and 2) the nature draw the weak type with the probability $1 - p$, the weak type plays Q with the probability one (separating), and Player 2 plays NF with probability $\sigma_2(\varepsilon_1)$ after observing Q. Thus, the conditional probability of observing (Q, NF) conditional on $(\varepsilon_1, \varepsilon_2)$ is given by

$$(p(1 - \mu_B^s(\varepsilon_2)) + (1-p)) \cdot \sigma_2(\varepsilon_1) \tag{41}$$

and we obtain the unconditional probability by taking expectation of (41) with respect to $(\varepsilon_1, \varepsilon_2) \in \boldsymbol{S}_6$. Other probabilities can be interpreted similarly.

Some changes of variables using the relationship of μ_B^s and ε_2 and that of σ_2 and ε_1 as shown in (38) give us

$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{S}_6, \mu_B^s, \sigma_2) =$$
$$p \int_0^1 \mu_B^s \; g_2 \left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p}{(1-\mu_B^s)p + (1-p)} \right) (\phi_{2s} + \phi_{2w}) - \phi_{2w} \right) \frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(1-\mu_B^s p)^2} d\mu_B^s$$
$$\times \int_0^1 \sigma_2 g_1 \left(\mu_s + X_1\beta_s + \frac{\phi_{1s}}{1-\sigma_2} \right) \frac{\phi_{1s}}{(1-\sigma_2)^2} d\sigma_2$$

$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{S}_6, \mu_B^s, \sigma_2) =$$
$$p \int_0^1 \mu_B^s \; g_2 \left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p}{(1-\mu_B^s)p + (1-p)} \right) (\phi_{2s} + \phi_{2w}) - \phi_{2w} \right) \frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(1-\mu_B^s p)^2} d\mu_B^s$$
$$\times \int_0^1 (1 - \sigma_2) g_1 \left(\mu_s + X_1\beta_s + \frac{\phi_{1s}}{1-\sigma_2} \right) \frac{\phi_{1s}}{(1-\sigma_2)^2} d\sigma_2$$

$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{S}_6, \mu_B^s, \sigma_2) =$$
$$\int_0^1 (p(1 - \mu_B^s) + (1-p)) \; g_2 \left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p}{(1-\mu_B^s)p + (1-p)} \right) (\phi_{2s} + \phi_{2w}) - \phi_{2w} \right) \frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(1-\mu_B^s p)^2} d\mu_B^s$$
$$\times \int_0^1 \sigma_2 g_1 \left(\mu_s + X_1\beta_s + \frac{\phi_{1s}}{1-\sigma_2} \right) \frac{\phi_{1s}}{(1-\sigma_2)^2} d\sigma_2$$

$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{S}_6, \mu_B^s, \sigma_2)$$
$$\int_0^1 (p(1 - \mu_B^s) + (1-p)) \; g_2 \left(X_2'\beta_2 + \left(\frac{(1-\mu_B^s)p}{(1-\mu_B^s)p + (1-p)} \right) (\phi_{2s} + \phi_{2w}) - \phi_{2w} \right) \frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(1-\mu_B^s p)^2} d\mu_B^s$$
$$\times \int_0^1 (1 - \sigma_2) g_1 \left(\mu_s + X_1\beta_s + \frac{\phi_{1s}}{1-\sigma_2} \right) \frac{\phi_{1s}}{(1-\sigma_2)^2} d\sigma_2$$

where we use the fact that $d\varepsilon_2 = -\frac{p(1-p)}{(1-\mu_B^s p)^2}(\phi_{2s} + \phi_{2w}) d\mu_B^s$, $d\varepsilon_1 = \frac{\phi_{1s}}{(1-\sigma_2)^2} d\sigma_2$, and the independence of ε_1 and ε_2.

Similarly, in the equilibrium of **SS-2** under the region of $\boldsymbol{S}_2$, four observed outcomes can arise with following probabilities, respectively, denoted by

$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) = \int_{\boldsymbol{S}_2} (p + (1-p)\mu_B^w(\varepsilon_2)) \cdot \widetilde{\sigma}_2(\varepsilon_1) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2$$
$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) = \int_{\boldsymbol{S}_2} (p + (1-p)\mu_B^w(\varepsilon_2)) \cdot (1 - \widetilde{\sigma}_2(\varepsilon_1)) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2$$
$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) = \int_{\boldsymbol{S}_2} (1-p)(1 - \mu_B^w(\varepsilon_2)) \cdot \widetilde{\sigma}_2(\varepsilon_1) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2$$
$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) = \int_{\boldsymbol{S}_2} (1-p)(1 - \mu_B^w(\varepsilon_2)) \cdot (1 - \widetilde{\sigma}_2(\varepsilon_1)) g_1(\varepsilon_1) g_2(\varepsilon_2) d\varepsilon_1 d\varepsilon_2.$$

Some changes of variables using the relationship of μ_B^w and ε_2 and that of $\widetilde{\sigma}_2$ and ε_1 as shown in (39) give us

$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) =$$
$$\int_0^1 (p + (1-p)\mu_B^w)\, g_2\left(X_2'\beta_2 + \left(\frac{p}{p+\mu_B^w\cdot(1-p)}\right)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(p+(1-p)\mu_B^w)^2}\,d\mu_B^w$$
$$\times \int_0^1 \widetilde{\sigma}_2 g_1\left(\mu_w + X_1\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}\,d\widetilde{\sigma}_2$$

$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) =$$
$$\int_0^1 (p + (1-p)\mu_B^w)\, g_2\left(X_2'\beta_2 + \left(\frac{p}{p+\mu_B^w\cdot(1-p)}\right)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(p+(1-p)\mu_B^w)^2}\,d\mu_B^w$$
$$\times \int_0^1 (1-\widetilde{\sigma}_2) g_1\left(\mu_w + X_1\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}\,d\widetilde{\sigma}_2$$

$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) =$$
$$\int_0^1 (1-p)(1-\mu_B^w)\, g_2\left(X_2'\beta_2 + \left(\frac{p}{p+\mu_B^w\cdot(1-p)}\right)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(p+\mu_B^w\cdot(1-p))^2}\,d\mu_B^w$$
$$\times \int_0^1 \widetilde{\sigma}_2 g_1\left(\mu_w + X_1\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}\,d\widetilde{\sigma}_2$$

$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{S}_2, \mu_B^w, \widetilde{\sigma}_2) =$$
$$\int_0^1 (1-p)(1-\mu_B^w)\, g_2\left(X_2'\beta_2 + \left(\frac{p}{p+\mu_B^w\cdot(1-p)}\right)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\frac{p(1-p)(\phi_{2s}+\phi_{2w})}{(p+\mu_B^w\cdot(1-p))^2}\,d\mu_B^w$$
$$\times \int_0^1 (1-\widetilde{\sigma}_2) g_1\left(\mu_w + X_1\beta_w - \frac{\phi_{1w}}{\widetilde{\sigma}_2}\right)\frac{\phi_{1w}}{\widetilde{\sigma}_2^2}\,d\widetilde{\sigma}_2$$

where we use the fact that $d\varepsilon_2 = -\frac{p(1-p)}{\left(p+\mu_B^w(1-p)\right)^2}(\phi_{2s} + \phi_{2w})d\mu_B^w$, $d\varepsilon_1 = \frac{\phi_{1w}}{\widetilde{\sigma}_2^2}d\widetilde{\sigma}_2$, and the independence of ε_1 and ε_2.

D.3 Equilibrium Refinement and Uniqueness of Equilibrium with Asymmetric Payoffs

We make the following assumption to obtain uniqueness of equilibrium.

Condition 3 $\mu_w + X_1'\beta_w - \phi_{1w} < \mu_s + X_1'\beta_s + \phi_{1s}$ for all $X_1 \in \mathbb{S}(X_1)$.

Condition 4 $\mu_s + X_1'\beta_s - \phi_{1s} < \mu_w + X_1'\beta_w + \phi_{1w}$ for all $X_1 \in \mathbb{S}(X_1)$.

These conditions are sufficient for uniqueness of equilibria with the refinement of Cho and Kreps (1987). Note that Condition 3 holds immediately when $\beta_w = \beta_s$, and $\mu_s = \mu_w$ (symmetric payoffs) since $\phi_{1s}, \phi_{1w} > 0$. Note that Condition 3 ensures the existence of the separating equilibrium (B, Q) with $\left(Y_{2|B} = NF, Y_{2|Q} = F\right)$ (see S1-21 & S1-22 in Section D.2.2) and Condition 4 eliminates the separating equilibrium (Q, B) with $\left(Y_{2|B} = NF, Y_{2|Q} = F\right)$ (see S2-21 & S2-22 in Section D.2.2).

- Under region $\boldsymbol{A} \equiv \{(\varepsilon_1, \varepsilon_2)|\mu_s + X_1'\beta_s - \phi_{1s} < \varepsilon_1$ and $X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$, we show that the pooling (B, B) with $A_{2|B} = F$ cannot survive the Intuitive Criterion of Cho and Kreps (1987).

  - Pooling with (B, B) with $A_{2|B} = F$

    Note $u_1(t_s; B, F) > u_1(t_s; Q, NF)$ since $\mu_s + X_1'\beta_1 - \varepsilon_1 - \phi_{1s} < 0$ under $\boldsymbol{A}$ and note $u_1(t_s; B, F) > u_1(t_s; Q, F)$ since $\phi_{1s} > 0$. This means that the *strong* type has no incentive to deviate in any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|Q) = 0$ and hence Player 2 will choose F after observing the deviation play Q under $\boldsymbol{A}$. Now we need to check whether the *weak* type is better off by deviation under this situation. Note
    $$u_1(t_w; B, F) = -\phi_{1w} < u_1(t_w; Q, F) = 0.$$
    Thus, the *weak* type will deviate for sure. Therefore, the equilibrium outcome fails the Intuitive Criterion.

- Under region $\widetilde{\boldsymbol{A}} \equiv \{(\varepsilon_1, \varepsilon_2)|\mu_s + X_1'\beta_s + \phi_s < \varepsilon_1$ and $X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + \phi_{2s}\}$, we show that the pooling (Q, Q) with $A_{2|Q} = F$ survives the Intuitive Criterion of Cho and Kreps (1987).

  - Pooling with (Q, Q) with $A_{2|Q} = F$

    Note $u_1(t_w; Q, F) > u_1(t_w; B, NF)$ since $\mu_w + X_1'\beta_w - \varepsilon_1 - \phi_{1w} < 0$ under $\widetilde{\boldsymbol{A}}$ by Condition 3 and note $u_1(t_w; Q, F) > u_1(t_w; B, F)$ since $\phi_{1w} > 0$. This means that the *weak* type has no incentive to deviate in

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any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|B) = 1$ and hence Player 2 will choose NF after observing the deviation play B under region $\widetilde{\boldsymbol{A}}$. Now we need to check whether the *strong* type is better off by deviation under this situation. Note

$$u_1(t_s; Q, F) = -\phi_{1s} > u_1(t_s; B, NF) = \mu_s + X_1'\beta_s - \varepsilon_1$$

under $\widetilde{\boldsymbol{A}}$. Thus, the *strong* type will not deviate. Therefore, the equilibrium outcome survives the Intuitive Criterion.

- Under region $\boldsymbol{B} \equiv \{(\varepsilon_1, \varepsilon_2)|\mu_w + X_1'\beta_w + \phi_w > \varepsilon_1 \text{ and } X_2'\beta_2 - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}\}$, the pooling (Q, Q) with $A_{2|Q} = NF$ cannot survive the Intuitive Criterion of Cho and Kreps (1987).

  - Pooling with (Q, Q) with $A_{2|Q} = NF$

    Note $u_1(t_w; Q, NF) > u_1(t_w; B, F)$ since $\mu_w + X_1'\beta_w - \varepsilon_1 + \phi_{1w} > 0$ under $\boldsymbol{B}$ and note $u_1(t_w; Q, NF) > u_1(t_w; B, NF)$ since $\phi_{1w} > 0$. This means that the *weak* type has no incentive to deviate in any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|B) = 1$ and hence Player 2 will choose NF after observing the deviation play B under region $\boldsymbol{B}$. Now we need to check whether the *strong* type is better off by deviation under this situation. Note

    $$u_1(t_s; Q, NF) = \mu_s + X_1'\beta_s - \varepsilon_1 - \phi_{1s} < u_1(t_s; B, NF) = \mu_s + X_1'\beta_s - \varepsilon_1$$

    since $\phi_{1s} > 0$. Thus, the *strong* type will deviate for sure. Therefore, the equilibrium outcome fails the Intuitive Criterion.

- Under region $\widetilde{\boldsymbol{B}} \equiv \{(\varepsilon_1, \varepsilon_2)|\mu_w + X_1'\beta_w - \phi_{1w} > \varepsilon_1 \text{ and } X_2'\beta_2 - \phi_{2w} < \varepsilon_2 < X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}\}$, the pooling (B, B) with $A_{2|B} = NF$ survives the Intuitive Criterion of Cho and Kreps (1987).

  - Pooling (B, B) with $A_{2|B} = NF$

    Note $u_1(t_s; B, NF) > u_1(t_s; Q, NF)$ since $\phi_{1s} > 0$ and note $u_1(t_s; B, NF) > u_1(t_s; Q, F)$ under $\widetilde{\boldsymbol{B}}$ by Condition 3. This means that the *strong* type has no incentive to deviate in any case. Thus, Player 2 will assign $\mu_2(t_1 = t_s|Q) = 0$ and hence Player 2 will choose F after observing the deviation play Q under region $\widetilde{\boldsymbol{B}}$. Now we need to check whether the *weak* type is better off by deviation under this situation. Note

    $$u_1(t_w; B, NF) = \mu_w + X_1'\beta_w - \varepsilon_1 - \phi_w > u_1(t_w; Q, F) = 0$$

    under region $\widetilde{\boldsymbol{B}}$. Thus, the *weak* type will not deviate. Therefore, the equilibrium outcome survives the Intuitive Criterion.

E Alternative Information Structure (IS-2)

Here we introduce an additional incomplete information to the game. Namely, Player 2 has some private information on her payoff, not observed by Player 1.

Assumption E.1 *(IS-2)*

1. Player 1 knows its true type but Player 2 knows only the distribution of Player 1's types (i.e., p is known to Player 2).
2. The realizations of (X_1, ε_1) and (X_2, ε_2) are perfectly observed by Player 2 but Player 1 only observes (X_1, ε_1) and X_2. (X_1, X_2) are public information.
3. ε_1 and ε_2 are pure shocks observed by Player 2, but Player 1 only observe her own ε_1. They are independent of each other and of any other variables in the game. ε_1 is also independent of the type of Player 1.
4. Players' actions and beliefs constitute a Perfect Bayesian Equilibrium (Sequential Equilibrium). Whenever there exist multiple equilibria, only one equilibrium is chosen out of these according to some equilibrium refinements. Players are assumed to play actions and hold beliefs of this unique equilibrium.

Under the game with **IS**, both players can perfectly observe (X_1, ε_1) and (X_2, ε_2), respectively but under this game with **IS-2**, Player 1 cannot observe ε_2 while Player 2 observes (X_1, ε_1) and (X_2, ε_2) perfectly. It makes Player 1 take expectation over ε_2 when she derive her expected payoffs depending on her choice of signals under the game with **IS-2**.

E.1 Equilibrium of the Game

In Section E.2, we characterize all possible PBE and obtain regions of $(\varepsilon_1, \varepsilon_2)$ given X where a particular PBE exist. We use a simplifying notation $G_2^*(a) \equiv G_2\left(X_2'\beta_2 + a(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)$. Figure A3 summarizes the result. Figure A3 depicts the case that $\phi_{1w} > \phi_{1s}$ and $\frac{\phi_{1s}}{\phi_{1w}} > \frac{G_2^*(p) - G_2^*(0)}{G_2^*(1) - G_2^*(0)}$. However, none of these conditions are necessary to obtain a PBE. We, again, impose a structure $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$), which are innocuous since a meaningful signaling game requires that (i) each signal corresponds to a particular type of Player 1 and (ii) Player 2 has an incentive to single out a particular type.

We conclude that

Theorem E.1 *(Existence of Equilibrium under Additional Incomplete Information)*
*Suppose Assumptions **IS-2** and **SA-1** hold. Suppose also that $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). Then, there exist PBE for all regions of $(\varepsilon_1, \varepsilon_2)$ given X.*

See the Appendix E.2 for the proof. From Figure A3, we see that there exist multiple equilibria in several regions. Similarly to the case of **IS**, we determine which equilibrium survives the refinement but here we use Banks and Sobel (1987)'s *divinity* concept [37] instead, which is stronger than Cho and Kreps (1987)[38]. The results are summarized in the following theorem (see the Appendix E.3.1).

Theorem E.2 *Suppose Assumptions **IS-2** and **SA-1** hold and suppose that $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). Then, (i) The pooling equilibrium with $(A_{1s}, A_{1w}) = (B, B)$ survives the refinement of Banks and Sobel (1987) under $\varepsilon_1 \leq \mu + X_1'\beta_1 - \frac{\phi_{1w}}{G_2^*(p) - G_2^*(0)}$; (ii) The pooling equilibrium $(A_{1s}, A_{1w}) = (Q, Q)$ survives the refinement of Banks and Sobel (1987) under $\varepsilon_1 \geq \mu + X_1'\beta_1 + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(p)}$.*
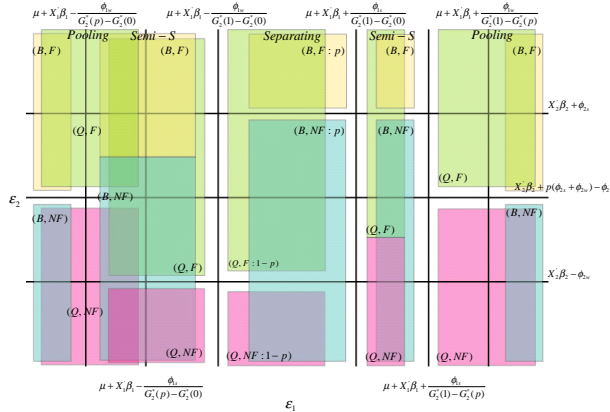

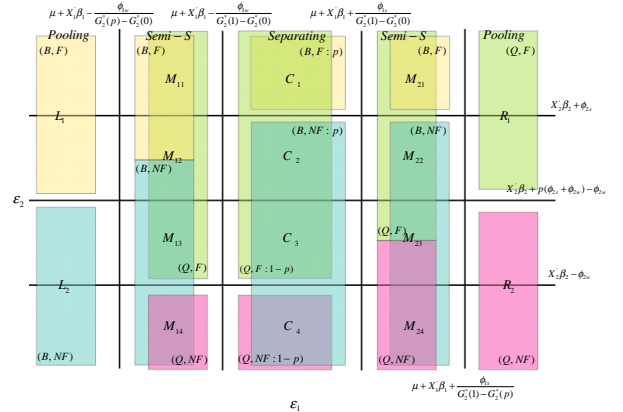FIGURE A3. Equilibrium of the Game under IS-2

FIGURE A4. Uniqueness of Equilibrium under IS-2

Figure A4 depicts the results and we conclude

Theorem E.3 *(Uniqueness of Equilibrium)*
*Suppose Assumptions **IS-2** and **SA-1** hold. Suppose $\phi_{1s} > 0$, $\phi_{1w} > 0$, $\phi_{2s} \geq 0$, and $\phi_{2w} \geq 0$ (but $\phi_{2s} \cdot \phi_{2w} \neq 0$). Further suppose each Player plays only one equilibrium that survives the refinement of Banks and Sobel (1987), when there exist multiple equilibria. Then, the game **G** has the unique equilibrium for each region of $(\varepsilon_1, \varepsilon_2)$ given X.*

---

[37] The idea of "divinity" developed by Banks and Sobel (1987) is that if Player 2 observes a deviation play and if the set of Player 2's responses that makes type t_1 willing to deviate from an equilibrium is strictly smaller than such a set for type t_1', it is reasonable to expect that Player 2 will think such deviation is more likely to be from type t_1'.

[38] We use Banks and Sobel (1987)'s refinement, instead of Cho and Kreps (1987), because it is not feasible to apply Cho and Kreps (1987)'s intuitive criterion to our game with additional incomplete information.

Now similarly with the case of **IS**, from the result of Theorem E.3, we can obtain a well-defined likelihood function under **IS-2** and **SA-1**. Using the conditional probabilities of four possible observed outcomes presented in the Appendix E.3.2, we can estimate the parameters of interest by the conditional ML as in (1). We note that the additional incomplete information considered in this section may help the identification of μ and other parameters since there are more critical values (four vertical lines in Figure A4 compared to two vertical lines in Figure 4) of ε_1 that switches the kinds of equilibrium and these critical lines vary according to different values of X_2. This is an interesting result.

E.2 PBE of the Game G with IS-2

Again we derive equilibria of the game with asymmetric payoffs as in Figure A1. Equilibria of the game with symmetric payoffs $(\mu_s = \mu_w, \beta_s = \beta_w)$ are easily obtained from the results in this section.

E.2.1 Pooling Equilibria

P-1) Pooling Equilibrium (both t_s and t_w choose B):

In this case, Player 2 does not update her belief and thus the posterior equals to the prior belief: $\mu_2(t_1 = t_s|B) = p$ and $\mu_2(t_1 = t_w|B) = 1 - p$. Thus, $Y_{2|B} = \mathbf{1}\{X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} - \varepsilon_2 \geq 0|B\}$ on the equilibrium path. Now we define $G_2^*(a) \equiv G_2\left(X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)$. The resulting payoffs of each type of Player 1 on-the-equilibrium path will be

	$Y_{2|B} = 1$	$Y_{2|B} = 0$	Player 1's expected payoffs
$u_1(t_s; B, \cdot)$	$\mu_s + X_1'\beta_s - \varepsilon_1$	0	$G_2^*(p)\left(\mu_s + X_1'\beta_s - \varepsilon_1\right)$
$u_1(t_w; B, \cdot)$	$\mu_w + X_1'\beta_w - \varepsilon_1 - \phi_{1w}$	$-\phi_{1w}$	$G_2^*(p)\left(\mu_w + X_1'\beta_w - \varepsilon_1\right) - \phi_{1w}$

To have the pooling $(A_{1t_s}, A_{1t_w}) = (B, B)$ as an equilibrium:

P1-1 t_s should have no incentive to deviate. Player 1 of t_s will note that Player 2 with $\overline{\mu}_2 \equiv \mu_2(t_1 = t_s|Q)$ as off-the-equilibrium belief will choose $Y_{2|Q} = \mathbf{1}\{X_2'\beta_2 + \overline{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w} - \varepsilon_2 \geq 0|Q\}$ which implies $\Pr(Y_{2|Q} = 1) = G_2\left(X_2'\beta_2 + \overline{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right) = G_2^*(\overline{\mu}_2)$. Thus, Player 1 of type t_s will have no incentive to deviate from the equilibrium if

$$E_1\left[u_1(t_s; Q, A_2)\right] \leq E_1[u_1(t_s; B, A_2)]$$

which implies

$$G_2^*(\overline{\mu}_2) \cdot \left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s} \leq G_2^*(p)\left(\mu_s + X_1'\beta_s - \varepsilon_1\right). \tag{42}$$

P1-2 t_w should have no incentive to deviate. t_w has an incentive to deviate unless $E_1\left[u_1(t_w; Q, A_2)\right] \leq E[u_1(t_w; B, A_2)]$. This requires

$$G_2^*(\overline{\mu}_2) \cdot \left(\mu_w + X_1'\beta_w - \varepsilon_1\right) \leq G_2^*(p)\left(\mu_w + X_1'\beta_w - \varepsilon_1\right) - \phi_{1w}. \tag{43}$$

Combining P1-1 and P1-2, we conclude that the pooling $(A_{1t_s}, A_{1t_w}) = (B, B)$ with $E\left[Y_{2|B}\right] = G_2^*(p)$ and $E\left[Y_{2|Q}\right] = G_2^*(\overline{\mu}_2)$ can be supported as an equilibrium under the following two cases.

- If $\overline{\mu}_2 > p$ (noting any $\overline{\mu}_2 > p$ satisfying (42) and (43) supports the equilibrium[39]), we need

$$\max\left\{\mu_s + X_1'\beta_s - \frac{\phi_{1s}}{G_2^*(1) - G_2^*(p)}, \mu_w + X_1'\beta_w + \frac{\phi_{1w}}{G_2^*(1) - G_2^*(p)}\right\} \leq \varepsilon_1. \tag{44}$$

- If $\overline{\mu}_2 < p$ (noting any $\overline{\mu}_2 < p$ satisfying (42) and (43) supports the equilibrium similarly as before), we need

$$\min\left\{\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(p) - G_2^*(0)}, \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(p) - G_2^*(0)}\right\} \geq \varepsilon_1. \tag{45}$$

---

[39] This means that there exist values of $\overline{\mu}_2 = \overline{\mu}_2(G_2^*(p), \mu_s + X_1'\beta_s, \mu_w + X_1'\beta_w, \phi_{1s}, \phi_{1w}, \varepsilon_1)$, $p < \overline{\mu}_2 \leq 1$ such that (42) and (43) hold.

P-2) Pooling Equilibrium (both t_s and t_w choose Q):

In this case, Player 2 does not update its belief and thus the posterior equals to the prior belief: $\mu_2(t_1 = t_s|Q) = p$ and $\mu_2(t_1 = t_w|Q) = 1 - p$. Thus, $Y_{2|Q} = \mathbf{1}\{X_2'\beta_2 + p(\phi_{2s} + \phi_{2w}) - \phi_{2w} - \varepsilon_2 \geq 0\}$ on the equilibrium path. The resulting payoffs of each type of Player 1 on-the-equilibrium path will be

	$Y_{2|Q} = 1$	$Y_{2|Q} = 0$	Player 1's expected payoffs
$u_1(t_s; Q, \cdot)$	$\mu_s + X_1'\beta_s - \varepsilon_1 - \phi_{1s}$	$-\phi_{1s}$	$G_2^*(p)\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s}$
$u_1(t_w; Q, \cdot)$	$\mu_w + X_1'\beta_w - \varepsilon_1$	0	$G_2^*(p)\left(\mu_w + X_1'\beta_w - \varepsilon_1\right)$

To have the pooling $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ as an equilibrium:

P2-1 t_w should have no incentive to deviate. Player 1 of t_w will note that Player 2 with $\widetilde{\mu}_2 \equiv \mu_2(t_1 = t_s|B)$ as off-the-equilibrium belief will choose $Y_{2|B} = \mathbf{1}\{X_2'\beta_2 + \widetilde{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w} - \varepsilon_2 \geq 0|Q\}$ which means $\Pr(Y_{2|B} = 1) = G_2\left(X_2'\beta_2 + \widetilde{\mu}_2(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right) = G_2^*(\widetilde{\mu}_2)$. Thus, Player 1 of type t_w will have no incentive to deviate from the equilibrium if

$$E_1\left[u_1\left(t_w; B, A_2\right)\right] \leq E_1[u_1(t_w; Q, A_2)]$$

which implies

$$G_2^*(\widetilde{\mu}_2) \cdot \left(\mu_w + X_1'\beta_w - \varepsilon_1\right) - \phi_{1w} \leq G_2^*(p)\left(\mu_w + X_1'\beta_w - \varepsilon_1\right). \tag{46}$$

P2-2 t_s should have no incentive to deviate. t_s has an incentive to deviate unless $E_1\left[u_1(t_s, B, A_2)\right] \leq E\left[u_1(t_s, Q, A_2)\right]$. This requires

$$G_2^*(\widetilde{\mu}_2) \cdot \left(\mu_s + X_1'\beta_s - \varepsilon_1\right) \leq G_2^*(p)\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s}. \tag{47}$$

Combining P1-1 and P1-2, we conclude that the pooling $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ with $E\left[Y_{2|Q}\right] = G_2^*(p)$ and $E\left[Y_{2|B}\right] = G_2^*(\widetilde{\mu}_2)$ can be supported as an equilibrium under the following two cases.

- If $\widetilde{\mu}_2 > p$ (noting any $\widetilde{\mu}_2 > p$ satisfying (46) and (47) supports the equilibrium), we need

$$\max\left\{\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(p)}, \mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(1) - G_2^*(p)}\right\} \leq \varepsilon_1. \tag{48}$$

- If $\widetilde{\mu}_2 < p$ (noting any $\widetilde{\mu}_2 < p$ satisfying (46) and (47) supports the equilibrium), we need

$$\min\left\{\mu_s + X_1'\beta_s - \frac{\phi_{1s}}{G_2^*(p) - G_2^*(0)}, \mu_w + X_1'\beta_w + \frac{\phi_{1w}}{G_2^*(p) - G_2^*(0)}\right\} \geq \varepsilon_1. \tag{49}$$

E.2.2 Separating Equilibria

S-1) Separating Equilibrium (t_s choose B and t_w choose Q):

In the separating equilibrium, Player 2 has complete information once it observes the signal. Thus, $\mu_2(t_1 = t_s|B) = 1$ and $\mu_2(t_1 = t_s|Q) = 0$. This implies $Y_{2|B} \equiv \mathbf{1}\{X_2'\beta_2 + \phi_{2s} - \varepsilon_2 \geq 0\}$. Similarly upon seeing Q Player 2 will have $Y_{2|Q} = \mathbf{1}\{X_2'\beta_2 - \varepsilon_2 \geq \phi_{2w}\}$. The resulting payoffs of each type of Player 1 on-the-equilibrium path will be

	Player 1's expected payoffs
$E_1[u_1(t_s; B, \cdot)]:$	$G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right)$
$E_1[u_1(t_w; Q, \cdot)]:$	$G_2(X_2'\beta_2 - \phi_{2w})\left(\mu_w + X_1'\beta_w - \varepsilon_1\right)$

S1-1 t_s should have no incentive to deviate. It requires
$$E_1\left[u_1(t_s; B, \cdot)\right] = G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) \geq E_1\left[u_1(t_s; Q, \cdot)\right] = G_2(X_2'\beta_2 - \phi_{2w})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s}.$$

S1-2 t_w should have no incentive to deviate. It requires
$$E_1[u_1(t_w; Q, \cdot)] = G_2(X_2'\beta_2 - \phi_{2w})\left(\mu_w + X_1'\beta_w - \varepsilon_1\right) \geq E_1\left[u_1(t_w; B, \cdot)\right] = G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_s + X_1'\beta_w - \varepsilon_1\right) - \phi_{1w}.$$

Combining S1-1 and S1-2, we conclude a separating equilibrium $(A_{1t_s}, A_{1t_w}) = (B, Q)$ can be supported as an equilibrium under[40]

$$\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)} \leq \varepsilon_1 \leq \mu_s + X_1'\beta_1 + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)}. \tag{50}$$

---

[40] If $\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)} > \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)}$, this equilibrium does not exist.

S-2) Separating Equilibrium (t_s choose Q and t_w choose B):

In the separating equilibrium, Player 2 has complete information once it observes the signal. Thus, $\mu_2(t_1 = t_s|B) = 0$ and $\mu_2(t_1 = t_w|B) = 1$. This implies $Y_{2|B} \equiv \mathbf{1}\left\{X_2'\beta_2 - \varepsilon_2 \geq \phi_{2w}\right\}$. Similarly upon seeing Q Player 2 will have $Y_{2|Q} = \mathbf{1}\left\{X_2'\beta_2 - \varepsilon_2 + \phi_{2s} \geq 0\right\}$. The resulting payoffs of each type of Player 1 on-the-equilibrium path will be

$$
\begin{array}{cl}
& \text{Player 1's expected payoffs} \\
E_1[u_1(t_s; Q, \cdot)]: & G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s} \\
E_1[u_1(t_w; B, \cdot)]: & G_2(X_2'\beta_2 - \phi_{2w})\left(\mu_w + X_1'\beta_w - \varepsilon_1\right) - \phi_{1w}
\end{array}
$$

S2-1 t_s should have no incentive to deviate. It requires
$$E_1[u_1(t_s; Q, \cdot)] = G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s} \geq E_1[u_1(t_s; B, \cdot)] = G_2(X_2'\beta_2 - \phi_{2w})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right).$$
S2-2 t_w should have no incentive to deviate. It requires
$$E_1[u_1(t_w; B, \cdot)] = G_2(X_2'\beta_2 - \phi_{2w})\left(\mu_w + X_1'\beta_w - \varepsilon_1\right) - \phi_{1w} \geq E_1[u_1(t_w; Q, \cdot)] = G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_w + X_1'\beta_w - \varepsilon_1\right).$$

Combining S2-1 and S2-2, we conclude a separating equilibrium $(A_{1t_s}, A_{1t_w}) = (Q, B)$ can be supported as an equilibrium under[41]

$$
\mu_w + X_1'\beta_w + \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)} \leq \varepsilon_1 \text{ and } X_1'\beta_s - \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)} \geq \varepsilon_1. \tag{51}
$$

E.2.3 Semi-separating Equilibrium

There exist other possible equilibria named "semi-separating" where one type of Player 1 randomize between two possible actions while the other type of Player 1 plays a separating equilibrium. One of conditions for the existence of such an equilibrium is that the type of Player 1 who randomizes should be indifferent between two actions.

SS1: **The weak type of Player 1 plays the separating equilibrium with Q and the strong type randomizes with $\sigma_{t_s}(B) = \mu_B^s$:**

Then,

$$
\mu_2(t_1 = t_s|B) = \frac{\mu_B^s \cdot p}{\mu_B^s \cdot p} = 1 \text{ and } \mu_2^{ss} \equiv \mu_2(t_1 = t_s|Q) = \frac{(1 - \mu_B^s)p}{(1 - \mu_B^s)p + (1 - p)} \in (0, p)
$$

which implies that

$$
\begin{array}{l}
E_1[u_1(t_s; B, \cdot)] = G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) \text{ and} \\
E_1[u_1(t_s; Q, \cdot)] = G_2\left(X_2'\beta_2 + \mu_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s}.
\end{array}
$$

For the existence of such a semi-separating equilibrium, it is required that the strong type of Player 1 is indifferent between choosing B or Q, $E_1[u_1(t_s; B, \cdot)] = E_1[u_1(t_s; Q, \cdot)]$ and hence
$G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) = G_2\left(X_2'\beta_2 + \mu_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\left(\mu_s + X_1'\beta_s - \varepsilon_1\right) - \phi_{1s}$
which implies

$$
\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2\left(X_2'\beta_2 + \mu_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)} = \varepsilon_1. \tag{52}
$$

We note that such μ_2^{ss} and ε_1 satisfying (52) does exist. Now we check whether the weak type of Player 1 has an incentive to deviate. The weak type has no incentive deviate if $E_1[u_1(t_w; Q, \cdot)] \geq E_1[u_1(t_w; B, \cdot)]$,
$G_2\left(X_2'\beta_2 + \mu_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\left(\mu_w + X_1'\beta_w - \varepsilon_1\right) \geq G_2(X_2'\beta_2 + \phi_{2s})\left(\mu_w + X_1'\beta_w - \varepsilon_1\right) - \phi_{1w}$
which holds as long as $\varepsilon_1 \geq \mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2\left(X_2'\beta_2 + \mu_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)}$. Note that by construction, we have $0 < \mu_2^{ss} < p$ and hence (52) implies that under the region of $\boldsymbol{M}_2$, such semi-separating equilibria exist.

SS2: **The strong type of Player 1 plays the separating equilibrium with B and the weak type randomizes with $\sigma_{t_w}(B) = \mu_B^w$:**

---

[41]No such ε_1 exists when $\mu_w + X_1'\beta_w + \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)} > \mu_s + X_1'\beta_s - \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)}$.

Then,

$$\widetilde{\mu}_2^{ss} = \mu_2(t_1 = t_s|B) = \frac{p}{p + \mu_B^w \cdot (1-p)} \in (p,1) \text{ and } \mu_2(t_1 = t_s|Q) = \frac{0}{(1-\mu_B^w)\cdot(1-p)} = 0$$

which implies that

$$E_1\left[u_1\left(t_w; B, \cdot\right)\right] = G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w})(\mu_w + X_1'\beta_w - \varepsilon_1) - \phi_{1w} \text{ and}$$
$$E_1\left[u_1\left(t_w; Q, \cdot\right)\right] = G_2\left(X_2'\beta_2 - \phi_{2w}\right)(\mu_w + X_1'\beta_w - \varepsilon_1).$$

For the existence of such a semi-separating equilibrium, it is required that the weak type of Player 1 is indifferent between choosing B or Q, $E_1\left[u_1\left(t_w; B, \cdot\right)\right] = E_1\left[u_1\left(t_w; Q, \cdot\right)\right]$ and hence
$G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w})(\mu_w + X_1'\beta_w - \varepsilon_1) - \phi_{1w} = G_2\left(X_2'\beta_2 - \phi_{2w}\right)(\mu_w + X_1'\beta_w - \varepsilon_1)$
which implies

$$\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}) - G_2\left(X_2'\beta_2 - \phi_{2w}\right)} = \varepsilon_1. \tag{53}$$

Now we check whether the strong type of Player 1 has an incentive to deviate. The strong type has no incentive deviate if $E_1\left[u_1\left(t_s; B, \cdot\right)\right] \geq E_1\left[u_1\left(t_s; Q, \cdot\right)\right]$,
$G_2\left(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)(\mu_s + X_1'\beta_s - \varepsilon_1) \geq G_2\left(X_2'\beta_2 - \phi_{2w}\right)(\mu_s + X_1'\beta_s - \varepsilon_1) - \phi_{1s}$
which holds as long as $\varepsilon_1 \leq \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2\left(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right) - G_2(X_2'\beta_2 - \phi_{2w})}$. Note that by construction, we have $p < \widetilde{\mu}_2^{ss} < 1$ and hence (53) implies that under the region of $\boldsymbol{M}_1$, such semi-separating equilibria exist.

E.2.4   Existence of Well-defined Likelihood for Semi-Separating

Under the equilibrium of (**SS1**), four observed outcomes can arise with following probabilities denoted by

$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{M}_2, \mu_B^s) = \Pr\left(Y_1 = 1; \boldsymbol{M}_2, \mu_B^s\right) \Pr\left(Y_2 = 1 | Y_1 = 1; ; \boldsymbol{M}_2, \mu_B^s\right)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_2} p \cdot \mu_B^s(\varepsilon_1) \cdot G_2(X_2'\beta_2 + \phi_{2s}) g_1(\varepsilon_1) d\varepsilon_1$$
$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{M}_2, \mu_B^s) = \Pr\left(Y_1 = 1; \boldsymbol{M}_2, \mu_B^s\right) \Pr\left(Y_2 = 0 | Y_1 = 1; \boldsymbol{M}_2, \mu_B^s\right)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_2} p \cdot \mu_B^s(\varepsilon_1) \cdot (1 - G_2(X_2'\beta_2 + \phi_{2s})) g_1(\varepsilon_1) d\varepsilon_1$$
$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{M}_2, \mu_B^s) = \Pr\left(Y_1 = 0; \boldsymbol{M}_2, \mu_B^s\right) \Pr\left(Y_2 = 0 | Y_1 = 1; \boldsymbol{M}_2, \mu_B^s\right)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_2} (p\left(1 - \mu_B^s(\varepsilon_1)\right) + (1-p)) \cdot G_2\left(X_2'\beta_2 + \mu_2^{ss}(\varepsilon_1)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right) g_1(\varepsilon_1) d\varepsilon_1$$
$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{M}_2, \mu_B^s) = \Pr\left(Y_1 = 0; \boldsymbol{M}_2, \mu_B^s\right) \Pr\left(Y_2 = 0 | Y_1 = 0; \boldsymbol{M}_2, \mu_B^s\right)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_2} (p\left(1 - \mu_B^s(\varepsilon_1)\right) + (1-p)) \cdot (1 - G_2\left(X_2'\beta_2 + \mu_2^{ss}(\varepsilon_1)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)) g_1(\varepsilon_1) d\varepsilon_1$$

where $\mu_2^{ss}(\varepsilon_1) = \frac{\left(1 - \mu_B^s(\varepsilon_1)\right)\cdot p}{\left(1 - \mu_B^s(\varepsilon_1)\right)\cdot p + (1-p)}$. For example, $\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{M}_2, \mu_B^s)$ is obtained using the following facts. Conditional on $(\varepsilon_1, \varepsilon_2)$, (Q, F) is observed in two cases: 1) the nature draw the strong type with the probability p, the strong type plays Q with the probability $(1 - \mu_B^s(\varepsilon_1))$, and Player 2 plays F with probability $(1 - G_2\left(X_2'\beta_2 + \mu_2^{ss}(\varepsilon_1)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right))$ after observing Q and 2) the nature draw the weak type with the probability $1 - p$, the weak type plays Q with the probability 1 (separating), and Player 2 plays F with probability $(1 - G_2\left(X_2'\beta_2 + \mu_2^{ss}(\varepsilon_1)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right))$ after observing Q. Thus, the conditional probability of observing (Q, F) conditional on $(\varepsilon_1, \varepsilon_2)$ is given by

$$(p\left(1 - \mu_B^s(\varepsilon_1)\right) + (1-p)) \cdot \left(1 - G_2\left(X_2'\beta_2 + \mu_2^{ss}(\varepsilon_1)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)\right) \tag{54}$$

and we obtain the unconditional probability by taking expectation of (54) with respect to $\varepsilon_1 \in \boldsymbol{M}_2$. Other probabilities can be interpreted similarly. Using a change of variables from the relationship of μ_B^s and ε_1:

$$\varepsilon_1(\mu_B^s, p) \equiv \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2\left(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)} \tag{55}$$

as given in (52), we find

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$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{M}_2, \mu_B^s)$$
$$= \int_0^1 p \cdot \mu_B^s \cdot G_2(X_2'\beta_2 + \phi_{2s}) g_1(\varepsilon_1(\mu_B^s, p)) D(\mu_B^s, p) d\mu_B^s$$
$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{M}_2, \mu_B^s)$$
$$= \int_0^1 p \cdot \mu_B^s \cdot (1 - G_2(X_2'\beta_2 + \phi_{2s})) g_1(\varepsilon_1(\mu_B^s, p)) D(\mu_B^s, p) d\mu_B^s$$
$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{M}_2, \mu_B^s)$$
$$= \int_0^1 (p(1 - \mu_B^s) + (1 - p)) \cdot G_2(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) g_1(\varepsilon_1(\mu_B^s, p)) D(\mu_B^s, p) d\mu_B^s$$
$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{M}_2, \mu_B^s)$$
$$= \int_0^1 (p(1 - \mu_B^s) + (1 - p)) \cdot (1 - G_2(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w})) g_1(\varepsilon_1(\mu_B^s, p)) D(\mu_B^s, p) d\mu_B^s$$

where $\mu_2^{ss}(\mu_B^s, p) = \frac{(1-\mu_B^s) \cdot p}{(1-\mu_B^s) \cdot p + (1-p)}$, $\varepsilon_1(\mu_B^s, p)$ as given in (55), and

$$D(\mu_B^s, p) \equiv \left| \frac{d\varepsilon_1(\mu_B^s, p)}{d\mu_B^s} \right| = \frac{\phi_{1s}(\phi_{2s} + \phi_{2w})}{(G_2(X_2'\beta_2 + \phi_{2s}) - G_2(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w}))^2}$$
$$\times g_2(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) \frac{p(1-p)}{((1-\mu_B^s)p + (1-p))^2}.$$

Similarly, under this equilibrium of (**SS2**), four observed outcomes can arise with following probabilities.

$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{M}_1, \mu_B^w) = \Pr(Y_1 = 1; \boldsymbol{M}_1, \mu_B^w) \Pr(Y_2 = 1 | Y_1 = 1; \boldsymbol{M}_1, \mu_B^w)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_1} (p + \mu_B^w(\varepsilon_1)(1 - p)) G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\varepsilon_1)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) g_1(\varepsilon_1) d\varepsilon_1$$
$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{M}_1, \mu_B^w) = \Pr(Y_1 = 1; \boldsymbol{M}_1, \mu_B^w) \Pr(Y_2 = 0 | Y_1 = 1; \boldsymbol{M}_1, \mu_B^w)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_1} (p + \mu_B^w(\varepsilon_1)(1 - p)) (1 - G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\varepsilon_1)(\phi_{2s} + \phi_{2w}) - \phi_{2w})) g_1(\varepsilon_1) d\varepsilon_1$$
$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{M}_1, \mu_B^w) = \Pr(Y_1 = 0; \boldsymbol{M}_1, \mu_B^w) \Pr(Y_2 = 0 | Y_1 = 1; \boldsymbol{M}_1, \mu_B^w)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_1} (1 - \mu_B^w(\varepsilon_1)) (1 - p) G_2(X_2'\beta_2 - \phi_{2w}) g_1(\varepsilon_1) d\varepsilon_1$$
$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{M}_1, \mu_B^w) = \Pr(Y_1 = 0; \boldsymbol{M}_1, \mu_B^w) \Pr(Y_2 = 0 | Y_1 = 0; \boldsymbol{M}_1, \mu_B^w)$$
$$= \int_{\varepsilon_1 \in \boldsymbol{M}_1} (1 - \mu_B^w(\varepsilon_1)) (1 - p) (1 - G_2(X_2'\beta_2 - \phi_{2w})) g_1(\varepsilon_1) d\varepsilon_1$$

where $\widetilde{\mu}_2^{ss}(\varepsilon_1) = \frac{p}{p + \mu_B^w(\varepsilon_1) \cdot (1-p)}$. A change of variables using the relationship of μ_2^{ss} and ε_1:

$$\varepsilon_1(\mu_B^w, p) = \mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\mu_B^w, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) - G_2(X_2'\beta_2 - \phi_{2w})} \tag{56}$$

as given in (53) gives us

$$\Pr(Y_1 = 1, Y_2 = 1; \boldsymbol{M}_1, \mu_B^w)$$
$$= \int_0^1 (p + \mu_B^w(1 - p)) G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\mu_B^w, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) g_1(\varepsilon_1(\mu_B^w, p)) D(\mu_B^w, p) d\mu_B^w$$
$$\Pr(Y_1 = 1, Y_2 = 0; \boldsymbol{M}_1, \mu_B^w)$$
$$= \int_0^1 (p + \mu_B^w(1 - p)) (1 - G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\mu_B^w, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w})) g_1(\varepsilon_1(\mu_B^w, p)) D(\mu_B^w, p) d\mu_B^w$$
$$\Pr(Y_1 = 0, Y_2 = 1; \boldsymbol{M}_1, \mu_B^w)$$
$$= \int_0^1 (1 - \mu_B^w) (1 - p) G_2(X_2'\beta_2 - \phi_{2w}) g_1(\varepsilon_1(\mu_B^w, p)) D(\mu_B^w, p) d\mu_B^w$$
$$\Pr(Y_1 = 0, Y_2 = 0; \boldsymbol{M}_1, \mu_B^s)$$
$$= \int_0^1 (1 - \mu_B^w) (1 - p) (1 - G_2(X_2'\beta_2 - \phi_{2w})) g_1(\varepsilon_1(\mu_B^w, p)) D(\mu_B^w, p) d\mu_B^w$$

where $\widetilde{\mu}_2^{ss}(\mu_B^w, p) = \frac{p}{p + \mu_B^w \cdot (1-p)}$, $\varepsilon_1(\mu_B^w, p)$ as given in (56), and

$$D(\mu_B^w, p) \equiv \left| \frac{d\varepsilon_1(\mu_B^w, p)}{d\mu_B^w} \right| = \frac{\phi_{1w}(\phi_{2s} + \phi_{2w})}{(G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\mu_B^w, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) - G_2(X_2'\beta_2 - \phi_{2w}))^2}$$
$$\times g_2(X_2'\beta_2 + + \widetilde{\mu}_2^{ss}(\mu_B^w, p)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) \frac{p(1-p)}{(p + \mu_B^w(1-p))^2}.$$

E.3   Existence of Well-defined Likelihood for the Game with IS-2

We proceed the following discussion for the game with public signals. All the discussions hold true for the game without public signals also by replacing $p(Z)$ with p. We add the public signal assumption to **IS-2**.

Assumption E.2 (IS-2A)

1 Assumption **IS-2** holds.
2 The public signal Z about the types of Player 1 is perfectly known to both Player 1 and Player 2.

E.3.1 Equilibrium Refinement and Uniqueness of Equilibrium (Proof of Theorem E.2)

Again we proceed our discussion with asymmetric payoffs case. From the result we obtain here, uniqueness of equilibrium for the symmetric payoffs case immediately follows. We make the following assumptions to obtain uniqueness of equilibrium.

Condition 5 $\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)} < \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)}$ *for all* $X_1 \times X_2 \in$
$\mathbb{S}(X_1) \times \mathbb{S}(X_2)$.

Condition 6 *For all* $W \in \mathbb{S}(W)$,
$$\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2\left(X_2'\beta_2+p(Z)(\phi_{2s}+\phi_{2w})-\phi_{2w}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)}$$
$$< \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2\left(X_2'\beta_2+p(Z)(\phi_{2s}+\phi_{2w})-\phi_{2w}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)}$$

Condition 7 *For all* $W \in \mathbb{S}(W)$,
$$\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2+p(Z)(\phi_{2s}+\phi_{2w})-\phi_{2w}\right)}$$
$$< \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2+p(Z)(\phi_{2s}+\phi_{2w})-\phi_{2w}\right)}$$

Condition 8 $\mu_s + X_1'\beta_s - \frac{\phi_{1s}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)} < \mu_w + X_1'\beta_w + \frac{\phi_{1w}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)}$ *for all* $X_1 \times X_2 \in$
$\mathbb{S}(X_1) \times \mathbb{S}(X_2)$.

These conditions are sufficient for uniqueness of equilibria together with the refinement of Banks and Sobel (1987), namely, *divine equilibrium*. Note that Conditions 5-8 hold immediately when $\mu_s = \mu_w$ and $\beta_w = \beta_s$ since $\phi_{1s}, \phi_{1w} > 0$. Condition 5 makes the separating (B, Q) supported (see (50)) and thus prevents two different semi-separating equilibria from overlapping each other (see Figure A4). Condition 8 eliminates the separating equilibrium (Q, B) (see (51)). Now we eliminate some of pooling equilibria using the refinement.

- Pooling with $(A_{1t_s}, A_{1t_w}) = (B, B)$

Under this pooling equilibrium, Player 2 knows the weak type is more willing to deviate (at any given Player 2's action) and hence Player 1 knows that if Player 2 observes a deviation play of Q, she will assign $\mu_2(t_1 = t_s|Q) < p$. In other words, it is reasonable to expect that the relative probability of t_s should decrease when Player 2 observes Q according to Banks and Sobel (1987)'s *divinity* concept. This rules out the case of (44) and only the region defined by (45) supports this equilibrium.

This refinement and Condition 6 make the pooling equilibrium with $(A_{1t_s}, A_{1t_w}) = (B, B)$ supported only under

$$\varepsilon_1 \le \mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2\left(X_2'\beta_2 + p(Z)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right) - G_2\left(X_2'\beta_2 - \phi_{2w}\right)}.$$

- Pooling with $(A_{1t_s}, A_{1t_w}) = (Q, Q)$

Under this pooling equilibrium, Player 2 knows the strong type is more willing to deviate (at any given Player 2's action) and hence Player 1 knows that if Player 2 observes a deviation play of B, she will assign $\mu_2(t_1 = t_s|B) > p$. Again Banks and Sobel (1987)'s *divinity* requires that we should expect that the relative probability of t_s should increase when Player 2 observes a deviation play with B. This rules out the case of (49) and only the region defined by (48) supports this equilibrium.

This refinement and Condition 7 make the pooling with equilibrium $(A_{1t_s}, A_{1t_w}) = (Q, Q)$ supported only under

$$\varepsilon_1 \ge \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2\left(X_2'\beta_2 + \phi_{2s}\right) - G_2\left(X_2'\beta_2 + p(Z)(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right)}.$$

Finally, we note that the estimation strategies considered in the paper are still valid based on the conditional probabilities derived in the next section provided that Assumption **SA-Asym-IS-2** holds.

Assumption E.3 (*SA-Asym-IS-2*)
(i) $\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)} < \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2\left(X_2'\beta_2+\phi_{2s}\right)-G_2\left(X_2'\beta_2-\phi_{2w}\right)}$ *for all* $X_1 \times X_2 \in \mathbb{S}(X_1) \times$
$\mathbb{S}(X_2)$ *and for all* $\mu_s, \mu_w, \phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w}, \beta_s, \beta_w, \beta_2$ *in the parameter space;*

(ii) $\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2(X_2'\beta_2 + (2p(Z)-1)\phi_2) - G_2(X_2'\beta_2 - \phi_{2w})} < \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2(X_2'\beta_2 + p(Z)(\phi_{2s} + \phi_{2w}) - \phi_{2w}) - G_2(X_2'\beta_2 - \phi_{2w})}$ *for all $W \in \mathbb{S}(W)$ and for all μ_s, μ_w, ϕ_{1s}, ϕ_{1w}, ϕ_{2s}, ϕ_{2w}, β_s, β_w, β_2, and $h(\cdot)$ in the parameter space;*

(iii)
$$\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2(X_2'\beta_2 + p(Z)(\phi_{2s} + \phi_{2w}) - \phi_{2w})}$$
$$< \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2(X_2'\beta_2 + p(Z)(\phi_{2s} + \phi_{2w}) - \phi_{2w})}$$

for all $W \in \mathbb{S}(W)$ and for all μ_s, μ_w, ϕ_{1s}, ϕ_{1w}, ϕ_{2s}, ϕ_{2w}, β_s, β_w, β_2, and $h(\cdot)$ in the parameter space;

(iv) $\mu_s + X_1'\beta_s - \frac{\phi_{1s}}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2(X_2'\beta_2 - \phi_{2w})} < \mu_w + X_1'\beta_w + \frac{\phi_w}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2(X_2'\beta_2 - \phi_{2w})}$ *for all $X_1 \times X_2 \in \mathbb{S}(X_1) \times \mathbb{S}(X_2)$ and for all μ_s, μ_w, ϕ_{1s}, ϕ_{1w}, ϕ_{2s}, ϕ_{2w}, β_s, β_w, β_2 in the parameter space.*

Note that Assumption **SA-Asym-IS-2** holds immediately when $\mu_s = \mu_w$ and $\beta_s = \beta_w$.

E.3.2 Conditional Probabilities of Four Observed Outcomes

From the result of previous sections, here we present the conditional probabilities of four observed outcomes in the game with **IS-2A** and **SA-2** allowing for asymmetric payoffs. For the game with **IS-2** and **SA-1**, we obtain the same conditional probabilities in replace of $p(Z)$ with p and W with X, respectively. Corresponding conditional probabilities of the game with symmetric payoffs are easily obtained by replacing β_s & β_w with β_1 and μ_s & μ_w with μ, respectively.

We will use some simplifying notations:

$$G_2^*(a) = G_2(X_2'\beta_2 + a(\phi_{2s} + \phi_{2w}) - \phi_{2w})$$
$$\mu_2^{ss}(\mu_B^s, p(Z)) = \frac{(1 - \mu_B^s) \cdot p(Z)}{(1 - \mu_B^s) \cdot p(Z) + (1 - p(Z))}$$
$$\varepsilon_1(\mu_B^s, p(Z)) = \mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2(X_2'\beta_2 + \phi_{2s}) - G_2(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s)(\phi_{2s} + \phi_{2w}) - \phi_{2w})}$$
$$D(\mu_B^s, p(Z)) \equiv \left| \frac{d\varepsilon_1(\mu_B^s, p(Z))}{d\mu_B^s} \right| = \frac{\phi_{1s}(\phi_{2s} + \phi_{2w})}{(G_2(X_2'\beta_2 + \phi_{2s}) - G_2(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s, p(Z))(\phi_{2s} + \phi_{2w}) - \phi_{2w}))^2}$$
$$\times g_2\left(X_2'\beta_2 + \mu_2^{ss}(\mu_B^s, p(Z))(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right) \frac{p(Z)(1 - p(Z))}{\left((1 - \mu_B^s)p(Z) + (1 - p(Z))\right)^2}$$
$$\widetilde{\mu}_2^{ss}(\mu_B^w, p(Z)) = \frac{p(Z)}{p(Z) + \mu_B^w \cdot (1 - p(Z))}$$
$$\widetilde{\varepsilon}_1(\mu_B^w, p(Z)) = \mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\mu_B^w, p(Z))(\phi_{2s} + \phi_{2w}) - \phi_{2w}) - G_2(X_2'\beta_2 - \phi_{2w})}$$
$$\widetilde{D}(\mu_B^w, p(Z)) \equiv \left| \frac{d\widetilde{\varepsilon}_1(\mu_B^w, p(Z))}{d\mu_B^w} \right| = \frac{\phi_{1w}(\phi_{2s} + \phi_{2w})}{(G_2(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\mu_B^w, p(Z))(\phi_{2s} + \phi_{2w}) - \phi_{2w}) - G_2(X_2'\beta_2 - \phi_{2w}))^2}$$
$$\times g_2\left(X_2'\beta_2 + \widetilde{\mu}_2^{ss}(\mu_B^w, p(Z))(\phi_{2s} + \phi_{2w}) - \phi_{2w}\right) \frac{p(Z)(1 - p(Z))}{\left(p(Z) + \mu_B^w(1 - p(Z))\right)^2}.$$

Now we present the conditional probabilities of four possible observed outcomes.

1 $(Y_1 = 1, Y_2 = 1) : (B, NF)$

It happens under region of $\boldsymbol{L}_2$ with probability one (pooling), under region of $\boldsymbol{C}_2 \cup \boldsymbol{C}_3 \cup \boldsymbol{C}_4$ (separating) with probability $p(Z)$, and under region of $\boldsymbol{M}_{12} \cup \boldsymbol{M}_{13} \cup \boldsymbol{M}_{14} \cup \boldsymbol{M}_{22} \cup \boldsymbol{M}_{23} \cup \boldsymbol{M}_{24}$ (semi-separating):

$$\Pr(Y_1 = 1, Y_2 = 1 | W, \alpha) =$$
$$G_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(p(Z)) - G_2^*(0)}\right) \cdot G_2^*(p(Z))$$
$$+ p(Z)\left(G_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)}\right) - G_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)}\right)\right) G_2^*(1)$$
$$+ \int_0^1 (p(Z) + \mu_B^w(1 - p(Z))) G_2^*(\widetilde{\mu}_2^{ss}(\mu_B^w, p(Z))) g_1(\widetilde{\varepsilon}_1(\mu_B^w, p(Z))) \widetilde{D}(\mu_B^w, p(Z)) d\mu_B^w$$
$$+ \int_0^1 p(Z)\mu_B^s G_2^*(1) g_1(\varepsilon_1(\mu_B^s, p(Z))) D(\mu_B^s, p(Z)) d\mu_B^s.$$

2 $(Y_1 = 1, Y_2 = 0) : (B, F)$

It happens under region of $\boldsymbol{L}_1$ with probability one (pooling) and under region of $\boldsymbol{C}_1$ (separating) with probability $p(Z)$, and under region of $\boldsymbol{M}_{11} \cup \boldsymbol{M}_{12} \cup \boldsymbol{M}_{21}$ (semi-separating):

$$\Pr(Y_1 = 1, Y_2 = 0 | W, \alpha) =$$
$$G_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(p(Z)) - G_2^*(0)}\right)(1 - G_2^*(p(Z)))$$
$$+ p(Z)\left(G_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)}\right) - G_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)}\right)\right)(1 - G_2^*(1))$$
$$+ \int_0^1 (p(Z) + \mu_B^w(1 - p(Z)))(1 - G_2^*(\widetilde{\mu}_2^{ss}(\mu_B^w, p(Z)))) g_1(\widetilde{\varepsilon}_1(\mu_B^w, p(Z))) \widetilde{D}(\mu_B^w, p(Z)) d\mu_B^w$$
$$+ \int_0^1 p(Z)\mu_B^s(1 - G_2^*(1)) g_1(\varepsilon_1(\mu_B^s, p(Z))) D(\mu_B^s, p(Z)) d\mu_B^s.$$

3 $(Y_1 = 0, Y_2 = 1) : (Q, NF)$

It happens under region of $\boldsymbol{R}_2$ (pooling) with probability one and under region of $\boldsymbol{C}_4$ (separating) with probability $1 - p(Z)$, and under region of $\boldsymbol{M}_{14} \cup \boldsymbol{M}_{23} \cup \boldsymbol{M}_{24}$ (semi-separating):

$$
\begin{aligned}
&\Pr(Y_1 = 0, Y_2 = 1|W, \alpha) = \\
&\left(1 - G_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(p(Z))}\right)\right) G_2^*\left(p(Z)\right) \\
&+ (1 - p(Z))\left(G_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)}\right) - G_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)}\right)\right) G_2^*\left(0\right) \\
&+ \int_0^1 (1 - \mu_B^w)(1 - p(Z)) G_2^*(0) g_1(\widetilde{\varepsilon}_1(\mu_B^w, p(Z))) \widetilde{D}(\mu_B^w, p(Z)) d\mu_B^w \\
&+ \int_0^1 (1 - p(Z)\mu_B^s) G_2^*(\mu_2^{ss}(\mu_B^s, p(Z))) g_1(\varepsilon_1(\mu_B^s, p(Z))) D(\mu_B^s, p(Z)) d\mu_B^s.
\end{aligned}
$$

4 $(Y_1 = 0, Y_2 = 0) : (Q, F)$

It happens under region of $\boldsymbol{R}_1$ (pooling) with probability one and under region of $\boldsymbol{C}_1 \cup \boldsymbol{C}_2 \cup \boldsymbol{C}_3$ (separating) with probability $1 - p(Z)$, and under region of $\boldsymbol{M}_{11} \cup \boldsymbol{M}_{12} \cup \boldsymbol{M}_{13} \cup \boldsymbol{M}_{21} \cup \boldsymbol{M}_{22} \cup \boldsymbol{M}_{23}$ (semi-separating):

$$
\begin{aligned}
&\Pr(Y_1 = 0, Y_2 = 0|W, \alpha) = \\
&\left(1 - G_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(p(Z))}\right)\right)\left(1 - G_2^*\left(p(Z)\right)\right) \\
&+ (1 - p(Z))\left(G_1\left(\mu_s + X_1'\beta_s + \frac{\phi_{1s}}{G_2^*(1) - G_2^*(0)}\right) - G_1\left(\mu_w + X_1'\beta_w - \frac{\phi_{1w}}{G_2^*(1) - G_2^*(0)}\right)\right)\left(1 - G_2^*(0)\right) \\
&+ \int_0^1 (1 - \mu_B^w)(1 - p(Z))(1 - G_2^*(0)) g_1(\widetilde{\varepsilon}_1(\mu_B^w, p(Z))) \widetilde{D}(\mu_B^w, p(Z)) d\mu_B^w \\
&+ \int_0^1 (1 - p(Z)\mu_B^s)(1 - G_2^*(\mu_2^{ss}(\mu_B^s, p(Z)))) g_1(\varepsilon_1(\mu_B^s, p(Z))) D(\mu_B^s, p(Z)) d\mu_B^s.
\end{aligned}
$$

F Mathematical Proofs for the Sieve Conditional ML

All the discussions of this section can be applied to both game models under **IS-A** and **IS-2A** based on appropriate conditional probabilities presented in Sections C and E.3.2, respectively.

F.1 Identification (Proof of Lemma 5.1)

Proof. We let $\mathcal{L}(Y, W, \alpha) = \exp(l(Y|W, \alpha))$ and note

$$
\frac{\mathcal{L}(Y, W, \alpha)}{\mathcal{L}(Y, W, \alpha_0)} \in \left\{ \frac{\Pr(Y = y|W, \alpha)}{\Pr(Y = y|W, \alpha_0)} : y \in \{(1,1), (1,0), (0,1), (0,0)\} \right\}
$$

where $\Pr(Y|W, \alpha)$ denotes the conditional probability of Y given $W = X \cup Z$ when the parameter equals to α. Applying Jensen's inequality, we have

$$
-\ln\left\{ E\left[\frac{\mathcal{L}(Y, W, \alpha)}{\mathcal{L}(Y, W, \alpha_0)}\right]\right\} < -E\left[\ln\left\{\frac{\mathcal{L}(Y, W, \alpha)}{\mathcal{L}(Y, W, \alpha_0)}\right\}\right] \tag{57}
$$

noting $\frac{\mathcal{L}(Y, W, \alpha)}{\mathcal{L}(Y, W, \alpha_0)}$ is always positive and not constant whenever $\alpha \neq \alpha_0$ by Assumption **SA-3**. We also have

$$
\Pr\left(\frac{\mathcal{L}(Y, W, \alpha)}{\mathcal{L}(Y, W, \alpha_0)} = \frac{\Pr(Y = y|W, \alpha)}{\Pr(Y = y|W, \alpha_0)}\right) = \Pr(Y = y|W, \alpha_0),
$$

for each $y \in \{(1,1), (1,0), (0,1), (0,0)\}$ under Assumptions **IS-A** and **SA-2** or Assumptions **IS-2A** and **SA-2**. It follows that

$$
\begin{aligned}
E\left[\frac{\mathcal{L}(Y, W, \alpha)}{\mathcal{L}(Y, W, \alpha_0)}\right] &= \int \left\{\sum_y \frac{\Pr(Y = y|W, \alpha)}{\Pr(Y = y|W, \alpha_0)} \cdot \Pr(Y = y|W, \alpha_0)\right\} f_W(W) dW \\
&= \int \left\{\sum_y \Pr(Y = y|W, \alpha)\right\} f_W(W) dW = 1
\end{aligned} \tag{58}
$$

where the last equality holds since $\sum_y \Pr(Y = y|W, \alpha) = 1$ for all $\alpha \in \mathcal{A}$. Therefore, combining (57) and (58), we conclude that for all $\alpha \neq \alpha_0 \in \mathcal{A}$, $0 < E[\ln \mathcal{L}(Y, W, \alpha_0)] - E[\ln \mathcal{L}(Y, W, \alpha)]$. This implies $Q(\alpha_0) > Q(\alpha)$ and thus proves the claim. ∎

F.2   Consistency

To prove the consistency, we need an additional condition. Recall that $\mathcal{L}_{ij}(W,\alpha)$ denotes the conditional probabilities of observed outcomes such that $\mathcal{L}_{ij}(W,\alpha) \equiv \Pr(Y_1 = i, Y_2 = j | W, \alpha)$ for $i, j = 0, 1$.

Condition 9 *(Lipschitz Condition)*
(i) For $\overline{\alpha}$ some convex combination of $\alpha_1, \alpha_2 \in \mathcal{A}_n$, there exists functions $M_{ij}^{(\phi_{1s})}(\cdot)$, $M_{ij}^{(\phi_{1w})}(\cdot)$, $M_{ij}^{(\phi_{2s})}(\cdot)$, $M_{ij}^{(\phi_{2w})}(\cdot)$, $M_{ij}^{(\mu)}(\cdot)$, $M_{ij}^{(\beta_1)}(\cdot)$, $M_{ij}^{(\beta_2)}(\cdot)$, and $M_{ij}^{(h)}(\cdot)$ such that

$$
\begin{aligned}
&\frac{d\mathcal{L}_{ij}(W,\overline{\alpha})}{d\alpha}[\alpha_1 - \alpha_2] = \\
&M_{ij}^{(\phi_{1s})}(W,\overline{\alpha})\,(\phi_{1s1} - \phi_{1s2}) + M_{ij}^{(\phi_{1w})}(W,\overline{\alpha})\,(\phi_{1w1} - \phi_{1w2}) + M_{ij}^{(\phi_{2s})}(W,\overline{\alpha})(\phi_{2s1} - \phi_{2s2}) \\
&+ M_{ij}^{(\phi_{2w})}(W,\overline{\alpha})(\phi_{2w1} - \phi_{2w2}) + M_{ij}^{(\mu)}(W,\overline{\alpha})(\mu_1 - \mu_2) \\
&+ M_{ij}^{(\beta_1)}(W,\overline{\alpha})X_1'\,(\beta_{11} - \beta_{12}) + M_{ij}^{(\beta_2)}(W,\overline{\alpha})X_2'\,(\beta_{21} - \beta_{22}) + M_{ij}^{(h)}(W,\overline{\alpha})\,(h_1 - h_2)
\end{aligned}
$$

for all $i, j = 0, 1$;
(ii) $\sup\limits_{\alpha \in \mathcal{A}_n} \left| M_{ij}^{(t)}(W,\alpha) \right| \leq C_t(W) < \infty$, $\forall t \in \{\phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w}, \mu, \beta_1, \beta_2, h\}$, $\forall i, j = 0, 1$, $n \geq \exists N$.

This condition is not difficult to verify from the arguments in Section J.

The following theorem is borrowed from Theorem 3.1 in Chen (2005) for the sieve conditional ML estimator defined in (11).

Theorem F.1 *(Theorem 3.1 in Chen (2005))*
Suppose (C1) $Q(\alpha)$ is uniquely maximized on $\mathcal{A}$ at $\alpha_0 \in \mathcal{A}$, and $Q(\alpha_0) > -\infty$;
(C2) $\mathcal{A}_n \subset \mathcal{A}_{n+1} \subset \mathcal{A}$ for all $n \geq 1$, and for any $\alpha \in \mathcal{A}$ there exists $\pi_n \alpha \in \mathcal{A}_n$ such that $\|\Pi_n \alpha - \alpha\|_s \to 0$ as $n \to \infty$;
(C3) The criterion function, $Q(\alpha)$, is continuous in $\alpha \in \mathcal{A}$ with respect to $\|\cdot\|_s$;
(C4) The sieve spaces, $\mathcal{A}_n$, are compact under $\|\cdot\|_s$;
(C5) $\plim\limits_{n\to\infty} \sup\limits_{\alpha \in \mathcal{A}_n} \left| \widehat{Q}_n(\alpha) - Q(\alpha) \right| = 0$ holds and let $\widehat{\alpha}_n$ be the approximate sieve ML estimator defined by (11), then $\|\widehat{\alpha}_n - \alpha\|_s = o_p(1)$.

Remarks (1)-(4) after Theorem 3.1 in Chen (2005) are applied here. Note that Condition (C1) is satisfied by Lemma 5.1. Condition (C2) holds for the sieve space $\mathcal{A}_n = \Theta \times \mathcal{H}_n$ with $\mathcal{H}_n$ defined in (8) (see Section 5.3.2 of Timan (1963)). Condition (C3) is satisfied since each $\mathcal{L}_{ij}$, $i, j = 1, 0$ is (pointwise) Lipschitz continuous by Condition 9 for the game models with **IS-A** or **IS-2A**. Condition (C4) holds for the sieve space (8). Now let $\mathcal{F}_n = \{l(y|w,\theta,h) : (\theta, h) \in \mathcal{A}_n\}$ denote the class of measurable functions indexed by (θ, h). Condition (C5) will be satisfied, for example, if $\mathcal{F}_n$ is P-Glivenko-Cantelli as presented in van der Vaart and Wellner (1996). The following lemma establishes the uniform convergence result.

Lemma F.1 *(Uniform convergence over sieves)*
*Suppose Assumptions **SA-2** and **SA-4** hold. Then, for $\widehat{Q}_n(\alpha)$ and $Q(\alpha)$ defined in (10) and (12), respectively and for $\mathcal{A}_n = \Theta \times \mathcal{H}_n$ with $\mathcal{H}_n$ defined in (8), we have $\plim\limits_{n\to\infty} \sup\limits_{\alpha \in \mathcal{A}_n} \left| \widehat{Q}_n(\alpha) - Q(\alpha) \right| = 0$.*

Proof. We prove this lemma by showing that all the conditions (i), (ii), and (iii) of Lemma A2 in Newey and Powell (2003) are satisfied. The condition (i) is satisfied for $\mathcal{A}_n = \Theta \times \mathcal{H}_n$ with $\mathcal{H}_n$ defined in (8) and for the metric $\|\cdot\|_s$. The condition (ii) will be satisfied if $E[|l(y_i|w_i, \theta, h(z_i))|] < \infty$, for all $(\theta, h) \in \mathcal{A}_n$ by the law of large numbers. Note that this condition is satisfied since $\mathcal{L}_{ij}(W, \theta, h)$, $\forall i, j = 0, 1$ is uniformly bounded between 0 and 1 over $\mathcal{A}_n$ and since under Assumptions **SA-2**,

$$
\Pr\left(\mathcal{L}_{ij}(W,\theta,h) = 0 \text{ or } \mathcal{L}_{ij}(W,\theta,h) = 1 \text{ for } W \in \mathbb{S}(W)\right) = 0 \text{ for } \forall i, j = 0, 1 \text{ and all } (\theta, h) \in \mathcal{A}_n,
$$

where $\Pr(\cdot)$ is a probability measure over W. Therefore, the condition (ii) holds. The condition (iii) is also satisfied since $\widehat{Q}_n(\alpha)$ is Lipschitz with respect to α by Condition 9. This completes the proof. ∎

Therefore, under Assumptions **SA-2**, **SA-3**, and **SA-4** and Condition 9, all the conditions in Theorem F.1 are satisfied for $\widehat{\alpha}_n$. This establishes the consistency result of the sieve ML estimator defined in (11).

F.3 Convergence Rate

In this section and the next section, we will use the following notations. For any $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{A}$, the pathwise first derivatives are defined as

$$\frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_2] \equiv \frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_0] - \frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\alpha_2 - \alpha_0] \text{ and}$$
$$\frac{dl(y_i|w_i,\alpha_3)}{d\alpha}[\alpha_1 - \alpha_0] \equiv \lim_{\tau \to 0} \frac{dl(y_i|w_i,\alpha_3+\tau(\alpha_1-\alpha_0))}{d\tau}.$$

The pathwise second derivatives are defined in a similar way such that

$$\frac{d^2 l(y_i|w_i,\alpha_3)}{d\alpha^2}[\alpha_1 - \alpha_0, \alpha_2 - \alpha_0] \equiv \lim_{\tau \to 0} \frac{dl(y_i|w_i,\alpha_3 + \tau(\alpha_2 - \alpha_0))}{d\alpha}[\alpha_1 - \alpha_0].$$

In particular, the pathwise second derivative at the direction $[\alpha - \alpha_0]$ evaluated at α_0 is denoted by $\frac{d^2 l(y_i|w_i,\alpha_0)}{d\alpha^2}[\alpha - \alpha_0, \alpha - \alpha_0] \equiv \lim_{\tau \to 0} \frac{d^2 l(y_i|w_i,\alpha_0+\tau(\alpha-\alpha_0))}{d\tau^2}$. Note that we have

$$E\left[\frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\alpha_1 - \alpha_0]\frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\alpha_2 - \alpha_0]\right] = -E\left[\frac{d^2 l(y_i|w_i,\alpha_0)}{d\alpha^2}[\alpha_1 - \alpha_0, \alpha_2 - \alpha_0]\right] \tag{59}$$

by Wong and Severini (1991).

Here we present a convergence rate of the sieve ML estimator defined in (11) under the metric $\|\cdot\|_2$. We use Theorem 3.2 of Chen (2005) which is a version of Chen and Shen (1996) for iid data. Before stating Theorem 3.2 of Chen (2005) and proving Proposition 5.1, we need to introduce some notations.

Now let $K(\alpha_0, \alpha) = n^{-1}\sum_{i=1}^n E\left[l(y_i|w_i,\alpha_0) - l(y_i|w_i,\alpha)\right]$ denote the Kullback-Leibler information (divergence measure) on n observations Let $\|\cdot\|_a$ be a metric on $\mathcal{A}$ which is equivalent to $K(\cdot,\cdot)^{1/2}$. The equivalence means there exist constants C_1 and $C_2 > 0$ such that $C_1 K(\alpha_0,\alpha)^{1/2} \leq \|\alpha_0 - \alpha\|_a \leq C_2 K(\alpha_0,\alpha)^{1/2}$ for all $\alpha \in \mathcal{A}$. It is well known in the literature that the convergence rate depends on two factors. One is how fast the sieve approximation error rate, $\|\alpha_0 - \Pi_n\alpha_0\|_a$, goes to zero. The other is the complexity of the sieve space $\mathcal{H}_n$. Let $L_r(P_0)$, $r \in [1,\infty)$ denote the space of real-valued random variables with finite r-th moments and $\|\cdot\|_{L_r}$ denote the $L_r(P_0)$-norm. Let $\mathcal{F}_n = \{g(\alpha,\cdot) : \alpha \in \mathcal{A}_n\}$ be a class of real-valued, $L_r(P_0)$-measurable functions indexed by $\alpha \in \mathcal{A}_n$. We let $N\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_r}\right)$ denote the *covering numbers without bracketing*, which is the minimal number of ε-balls $\left\{\{f : \|f - g_j\|_{L_r} \leq \varepsilon\}, \|g_j\|_{L_r} < \infty, j = 1,\ldots,N\right\}$ that covers $\mathcal{F}_n$. We often use the notion of $L_r(P_0)$-*metric entropy*, $H\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_r}\right) \equiv \log\left(N\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_r}\right)\right)$ as a measure of the complexity of $\mathcal{F}_n$ since the covering numbers can grow very fast. The second notion of complexity of the class $\mathcal{F}_n$ is the *covering numbers with bracketing*. Let $\mathcal{C}_{L_r}$ be the completion of $\mathcal{F}_n$ under the norm $\|\cdot\|_{L_r}$. The $L_r(P_0)$-*covering numbers with bracketing*, denoted by $N_{[]}\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_r}\right)$, is the minimal number of N for which there exist ε-brackets $\left\{[l_j, u_j] : l_j, u_j \in \mathcal{C}_{L_r}, \max_{1 \leq j \leq N}\|l_j - u_j\|_{L_r} \leq \varepsilon, \|l_j\|_{L_r} < \infty, \|u_j\|_{L_r} < \infty, j = 1,\ldots,N\right\}$ to cover $\mathcal{F}_n$ (i.e., for each $g \in \mathcal{F}_n$, there exists a $j = j(g) \in \{1,\ldots,N\}$ such that $l_j \leq g \leq u_j$ a.e.-P_0. Similarly we let $H_{[]}\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_r}\right) \equiv \log\left(N_{[]}\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_r}\right)\right)$ which is called the $L_r(P_0)$-*metric entropy with bracketing* of the class $\mathcal{F}_n$. Pollard (1984), Andrews (1994), van der Vaart and Wellner (1996), and van der Geer (2000) provide more detailed discussions of metric entropy. We will use a simplified notation $b_{1n} \asymp b_{2n}$ for two sequences of positive numbers b_{1n} and b_{2n} when the ratio of these two b_{1n}/b_{2n} is bounded below and above by some positive constants.

Now let $\mathcal{F}_n = \{l(y_i|w_i,\alpha) - l(y_i|w_i,\alpha_0) : \|\alpha - \alpha_0\|_a \leq \delta, \alpha \in \mathcal{A}_n\}$ and for some constant $b > 0$, let

$$\delta_n = \inf\left\{\delta \in (0,1) : \frac{1}{\sqrt{n}\delta^2}\int_{b\delta^2}^{\delta}\sqrt{H_{[]}\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_2}\right)}d\varepsilon \leq const\right\}. \tag{60}$$

Note that δ_n does not only depend on the smoothness of $l(\cdot|\cdot,\cdot)$ but also it depends on the complexity of the sieve $\mathcal{A}_n$. Now we present Theorem 3.2 in Chen (2005) tailored to our estimator.

Theorem F.2 *(Theorem 3.2 in Chen (2005))*
Suppose that (CC1) The data $\{Y_{1i}, Y_{2i}, W_i\}$ are iid;
(CC2) There is a $C_1 > 0$ such that for all small $\varepsilon > 0$, $\sup_{\alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\|_2 \leq \varepsilon} Var\left(l(y_i|w_i, \alpha) - l(y_i|w_i, \alpha_0)\right) \leq C_1 \varepsilon^2$;
(CC3) For any $\delta > 0$, there exists a constant $s \in (0, 2)$ such that
$\sup_{\alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\|_2 \leq \delta} |l(y_i|w_i, \alpha) - l(y_i|w_i, \alpha_0)| \leq \delta^s U(w_i)$ *with* $E\left[|U(w_i)|^t\right] \leq C_2$ *for some* $t \geq 2$. *Let* $\widehat{\alpha}_n$ *be the approximate sieve ML defined in (11). Then,* $\|\widehat{\alpha}_n - \alpha_0\|_2 = O_p(\varepsilon_n)$ *with* $\varepsilon_n = \max\{\delta_n, \|\Pi_n \alpha_0 - \alpha_0\|_2\}$.

Note that δ_n increases with the complexity of the sieve $\mathcal{A}_n$, which can be interpreted as a measure of the standard deviation form, while we interpret the deterministic approximation error $\|\Pi_n \alpha_0 - \alpha_0\|_2$ as a measure of the bias since it decreases with the complexity of the sieve $\mathcal{A}_n$. Now we prove Proposition 5.1 by showing the conditions in Theorem F.2 hold under Assumptions **SA-2**, **SA-3**, and **SA-4**. We impose the following three conditions

Condition 10 *For $\overline{\alpha}$ some convex combination of α and α_0 and for $\mathcal{L}_{ij}$, there exists functions $M_{ij}^{(\phi_{1s})}(\cdot)$, $M_{ij}^{(\phi_{1w})}(\cdot)$, $M_{ij}^{(\phi_{2s})}(\cdot)$, $M_{ij}^{(\phi_{2w})}(\cdot)$, $M_{ij}^{(\mu)}(\cdot)$, $M_{ij}^{(\beta_1)}(\cdot)$, $M_{ij}^{(\beta_2)}(\cdot)$, and $M_{ij}^{(h)}(\cdot)$ such that*

$$
\frac{d\mathcal{L}_{ij}(W, \overline{\alpha})}{d\alpha}[\alpha - \alpha_0] =
$$
$$
M_{ij}^{(\phi_{1s})}(W, \overline{\alpha})\left(\phi_{1s} - \phi_{1s0}\right) + M_{ij}^{(\phi_{1w})}(W, \overline{\alpha})\left(\phi_{1w} - \phi_{1w0}\right) + M_{ij}^{(\phi_{2s})}(W, \overline{\alpha})\left(\phi_{2s} - \phi_{2s0}\right) + M_{ij}^{(\phi_{2w})}(W, \overline{\alpha})\left(\phi_{2w} - \phi_{2w0}\right)
$$
$$
+ M_{ij}^{(\mu)}(W, \overline{\alpha})\left(\mu - \mu_0\right) + M_{ij}^{(\beta_1)}(W, \overline{\alpha})X_1'\left(\beta_1 - \beta_{10}\right) + M_{ij}^{(\beta_2)}(W, \overline{\alpha})X_2'\left(\beta_2 - \beta_{20}\right) + M_{ij}^{(h)}(W, \overline{\alpha})\left(h - h_0\right)
$$

for all $i, j = 0, 1$.

Condition 11 *(i)* $\sup_{\alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\|_s = o(1)} \left| M_{ij}^{(t)}(W, \alpha) \right| \leq C_{11}(W) < \infty$, $\forall t \in \{\phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w}, \mu, \beta_1, \beta_2, h\}$, $\forall i, j = 0, 1$;

(ii) $\inf_{\alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\|_s = o(1)} \left| M_{ij}^{(t)}(W, \alpha) \right| \geq C_{12}(W) > 0$, $\forall t \in \{\phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w}, \mu, \beta_1, \beta_2, h\}$, $\forall i, j = 0, 1$;

Condition 12 $\sup_{\alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\|_s = o(1)} \left| M_{ij}^{(t)}(W, \alpha) - M_{ij}^{(t)}(W, \alpha_0) \right| = C_2(W) \|\alpha - \alpha_0\|_s$ *with* $C_2(W) < \infty$,
$\forall t \in \{\phi_{1s}, \phi_{1w}, \phi_{2s}, \phi_{2w}, \mu, \beta_1, \beta_2, h\}$, $\forall i, j = 0, 1$.

These three conditions are not difficult to verify for the game models with **IS-A** and **IS-2A** from the arguments in Section J. These three conditions are sufficient to verify the conditions (CC2) and (CC3).

F.3.1 Proof of Proposition 5.1

The condition (CC1) is directly assumed by Assumption **SA-4** (i). Condition 10 implies that the pathwise derivative of $l(\cdot|\cdot, \cdot)$ is well defined. Condition 10-11 implies that $l(\cdot|\cdot, \cdot)$ satisfies a Lipschitz condition in $\alpha \in \mathcal{A}$. Condition 12 is useful to provide some regularities on the difference of the pathwise derivatives of $l(\cdot|\cdot, \cdot)$. We first verify the condition (CC2). Using the mean value theorem, we have $l(y_i|w_i, \alpha) - l(y_i|w_i, \alpha_0) = \frac{dl(y_i|w_i, \widetilde{\alpha})}{d\alpha}[\alpha - \alpha_0]$ where $\widetilde{\alpha}$ lies between α and α_0. It follows that

$$
E\left[\left(l(y_i|w_i, \alpha) - l(y_i|w_i, \alpha_0)\right)^2\right] = E\left[\left(\frac{dl(y_i|w_i, \widetilde{\alpha})}{d\alpha}[\alpha - \alpha_0]\right)^2\right]
$$

from which we conclude $E\left[\left(l(y_i|w_i, \alpha) - l(y_i|w_i, \alpha_0)\right)^2\right] \asymp \|\alpha - \alpha_0\|_2$ since Condition 10-12 implies that

$$
\left| \frac{dl(y_i|w_i, \widetilde{\alpha})}{d\alpha}[\alpha - \alpha_0] - \frac{dl(y_i|w_i, \alpha_0)}{d\alpha}[\alpha - \alpha_0] \right| = O(\|\alpha - \alpha_0\|_s^2) \text{ with } \|\alpha - \alpha_0\|_s = o(1).
$$

Thus, the condition (CC2) holds immediately.

Now we verify the condition (CC3). Note if $0 < C \leq a \leq b$, then $|\ln a - \ln b| \leq |a - b|/C$. Because $\mathcal{L}_{ij}(W, \alpha)$, $\forall i, j = 0, 1$ is bounded away from zero and one for all $\alpha \in \left\{\alpha \in \mathcal{A}_n : \|\alpha - \alpha_0\|_2 \leq \varepsilon\right\}$ for all small $\varepsilon > 0$, we have

$$
|\ln \mathcal{L}_{ij}(W, \alpha) - \ln \mathcal{L}_{ij}(W, \alpha_0)| \leq |\mathcal{L}_{ij}(W, \alpha) - \mathcal{L}_{ij}(W, \alpha_0)|/C \text{ for all } i, j = 0, 1.
$$

This fact and the fact that $\mathcal{L}_{ij}(W, \alpha)$ satisfies the Lipschitz condition for all $i, j = 0, 1$ (Condition 10-11) implies that

$$|l(Y|W, \alpha) - l(Y|W, \alpha_0)| \leq C(W) \|\alpha - \alpha_0\|_s \tag{61}$$

with $E\left[C(W)^2\right] < \infty$. By Theorem 1 of Gabushin (1967) (for integer $\nu_1 > 0$) or Lemma 2 in Chen and Shen (1998) (for $\nu_1 > 0$, any positive number), we have

$$\|\alpha - \alpha_0\|_s \leq C_1 \|\alpha - \alpha_0\|_2^{\frac{2\nu_1/d_z}{2\nu_1/d_z + 1}}. \tag{62}$$

From (61) and (62), we see that the condition (CC3) is satisfied with $s = 2\frac{\nu_1}{d_z}/(2\frac{\nu_1}{d_z} + 1)$ and $U(w_i) = C_1 C(W)$. We have verified all the conditions in Theorem F.2.

The next step is to derive the convergence rate depending on the choice of sieves. For the sieve $\mathcal{A}_n \equiv \Theta \times \mathcal{H}_n$ with $\mathcal{H}_n$ defined in (8), we have

$$\|\Pi_n \alpha_0 - \alpha_0\|_s = O\left(K_n^{-\nu_1/d_z}\right) \tag{63}$$

by Lorentz (1986). Now we need to calculate δ_n that solves (60). Now let $C = \sqrt{E\left[|U(w_i)|^2\right]}$ and $u_h \equiv \sup_{h \in \mathcal{H}_n} \|h\|_\infty$, then for all $0 < \frac{\varepsilon}{C} \leq \delta < 1$, we have

$$H_{[]}\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_2}\right) \leq \log N\left(\frac{\varepsilon}{C}, \mathcal{H}_n, \|\cdot\|_\infty\right) \leq const \times K_n \times \log\left(1 + \frac{4u_h}{\varepsilon}\right)$$

by Lemma 2.5 in van der Geer (2000). Using this result, we obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{n}\delta_n^2} \int_{b\delta_n^2}^{\delta_n} \sqrt{H_{[]}\left(\varepsilon, \mathcal{F}_n, \|\cdot\|_{L_2}\right)} d\varepsilon \leq C\frac{1}{\sqrt{n}\delta_n^2} \int_{b\delta_n^2}^{\delta_n} \sqrt{K_n \times \log\left(1 + \frac{4u_h}{\varepsilon}\right)} d\varepsilon \\
&\leq C\frac{1}{\sqrt{n}\delta_n^2} \sqrt{K_n} \int_{b\delta_n^2}^{\delta_n} \log\left(1 + \frac{4u_h}{\varepsilon}\right) d\varepsilon \leq C\frac{1}{\sqrt{n}\delta_n^2} \sqrt{K_n}\delta_n \leq const.
\end{aligned} \tag{64}
$$

From the last inequality of (64), we conclude $\delta_n \asymp \sqrt{\frac{K_n}{n}}$. We complete the proof by combining this result with (63). Finally, by letting $\delta_n \asymp \|\Pi_n \alpha_0 - \alpha_0\|_s$ and $K_n = n^\kappa$, we obtain the optimal rate with $K_n = n^{1/(2v_1/d_z + 1)}$.

F.4 Asymptotic Normality

We derive the asymptotic normality of the structural parameters estimates using Theorem 4.3 in Chen (2005).

First, we let the sieve conditional ML estimator $\widehat{\alpha}_n$ converges to α_0 at a rate faster than η_n. Also let ε_n denote any sequence satisfying $\varepsilon_n = o(n^{-1/2})$ and $\mu_n(g(Y, W)) = n^{-1} \sum_{i=1}^n \{g(y_i, w_i) - E[g(y_i, w_i)]\}$ denote the empirical process indexed by the function g. The following theorem show that the plug-in sieve conditional ML estimator $f(\widehat{\alpha}_n)$ achieves the $\sqrt{n}$-asymptotic normality.

Theorem F.3 *(Theorem 4.3 in Chen (2005))*
Suppose that (AN1) (i) there is an $\omega > 0$ such that $\left|f(\alpha) - f(\alpha_0) - \frac{df(\alpha_0)}{d\alpha}[\alpha - \alpha_0]\right| = O\left(\|\widehat{\alpha}_n - \alpha_0\|_2^\omega\right)$ uniformly over $\alpha \in \mathcal{A}_n$ with $\|\alpha - \alpha_0\|_2 = o(1)$; (ii) $\left\|\frac{df(\alpha_0)}{d\alpha}\right\| < \infty$; (iii) there is a $\Pi_n v^* \in \mathcal{A}_n$ such that $\|\Pi_n v^* - v^*\|_2 \times \|\widehat{\alpha}_n - \alpha_0\|_2 = o_p(n^{-1/2})$;
(AN2) $\sup_{\alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\|_2 \leq \eta_n} \mu_n\left(l(Y|W, \alpha) - l(Y|W, \alpha \pm \varepsilon_n \Pi_n v^*) - \frac{dl(Y|W, \alpha_0)}{d\alpha}[\pm \varepsilon_n \Pi_n v^*]\right) = O_p(\varepsilon_n^2)$;
(AN3) $K(\alpha_0, \widehat{\alpha}_n) - K(\alpha_0, \widehat{\alpha}_n \pm \varepsilon_n \Pi_n v^*) = \pm \varepsilon_n \times \langle \widehat{\alpha}_n - \alpha_0, \Pi_n v^* \rangle + o(n^{-1})$;
(AN4) (i) $\mu_n\left(\frac{dl(Y|W, \alpha_0)}{d\alpha}[\Pi_n v^* - v^*]\right) = o_p(n^{-1/2})$; (ii) $E\left[\frac{dl(Y|W, \alpha_0)}{d\alpha}[\Pi_n v^*]\right] = o(n^{-1/2})$;
(AN5) $n^{1/2}\mu_n\left(\frac{dl(Y|W, \alpha_0)}{d\alpha}[v^*]\right) \underset{d}{\to} N(0, \sigma_{v^*}^2)$ with $\sigma_{v^*}^2 \equiv Var\left[\frac{dl(Y|W, \alpha_0)}{d\alpha}[v^*]\right] > 0$ for iid data holds and $\|\widehat{\alpha}_n - \alpha_0\|_2^\omega = o_p(n^{-1/2})$. Then, for the sieve ML estimate $\widehat{\alpha}_n$ given in (14), we have $\sqrt{n}(f(\widehat{\alpha}_n) - f(\alpha_0)) \underset{d}{\to} N(0, \sigma_{v^*}^2)$.

Note that for statistical inference of the sieve plug-in estimate $f(\widehat{\alpha}_n)$, one needs a consistent estimator for $\sigma_{v^*}^2$. For example, such estimators can be found in Andrews (1994b), Newey (1994), and Ai and Chen (2003). Now we are ready to prove Theorem 5.1 based on Theorem F.3.

F.4.1 Proof of Theorem 5.1

Now note that $\frac{df(\alpha_0)}{d\alpha}[\alpha-\alpha_0] = (\theta-\theta_0)'\lambda$ which implies that $f(\alpha)-f(\alpha_0)-\frac{df(\alpha_0)}{d\alpha}[\alpha-\alpha_0] = 0$ and hence the condition (AN1) (i) holds with $\omega = \infty$. In addition, note

$$\sup_{0\neq\alpha-\alpha_0\in\overline{V}} \frac{|f(\alpha)-f(\alpha_0)|^2}{\|\alpha-\alpha_0\|^2} = \sup_{0\neq\alpha-\alpha_0\in\overline{V}} \frac{|\lambda'(\theta-\theta_0)|^2}{E\left[\left(\frac{dl(Y|W,\alpha_0)}{d\theta'}(\theta-\theta_0)+\frac{dl(Y|W,\alpha_0)}{dh}[h-h_0]\right)^2\right]}$$

$$= \sup_{0\neq(\theta-\theta_0,b)\in\overline{V}} \frac{\lambda'(\theta-\theta_0)(\theta-\theta_0)'\lambda}{(\theta-\theta_0)'E\left[\left(\frac{dl(Y|W,\alpha_0)}{d\theta'}-\frac{dl(Y|W,\alpha_0)}{dh}[b]\right)'\left(\frac{dl(Y|W,\alpha_0)}{d\theta'}-\frac{dl(Y|W,\alpha_0)}{dh}[b]\right)\right](\theta-\theta_0)}$$

$$= \lambda'\left(E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right]\right)^{-1}\lambda$$

which implies $f(\alpha) = \lambda'\theta$ is bounded if and only if $E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right]$ is finite positive-definite, in which case we have $v^* \in \overline{V}$ such that

$$f(\alpha) - f(\alpha_0) \equiv \lambda'(\theta-\theta_0) = \langle v^*, \alpha-\alpha_0\rangle \text{ for all } \alpha \in \mathcal{A} \tag{65}$$

by (18) and the Riesz representation theorem. $v^* \equiv (v_\theta^*, v_h^*) \in \overline{V}$ satisfies (65) with

$$v_\theta^* = \left(E\left[D_{b^*}(Y,W)'D_{b^*}(Y,W)\right]\right)^{-1}\lambda \text{ and } v_h^* = -b^* \times v_\theta^*.$$

Thus, the condition (AN1) (ii) is satisfied under Assumption **SA-5** (ii). Assumption **SA-5** (iii) implies that we can find $\Pi_n v^* \in \mathcal{A}_n \equiv \Theta \times \mathcal{H}_n$ with $\mathcal{H}_n$ defined in (8) such that $\|\Pi_n v^* - v^*\|_s = O\left(n^{-1/4}\right)$. Combining this with the condition $\|\widehat{\alpha}_n - \alpha_0\|_2 = o_p(n^{-1/4})$ supported by Proposition 5.1, we obtain $\|\Pi_n v^* - v^*\|_2 \times \|\widehat{\alpha}_n - \alpha_0\|_2 = o_p(n^{-1/2})$ with $\nu_1/d_z > 1/2$. This satisfies the condition (AN1) (iii). Next, we verify the condition (AN3).

Note that we have

$$E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha-\alpha_0]\right] = 0 \tag{66}$$

for any $\alpha - \alpha_0$ (it does not need to be in $\overline{V}$) because (i) the directional derivative of $l(Y|W,\alpha)$ at α_0 is well-defined and (ii) it is unconstrained maximization (see Shen (1997)). This is the zero expectation of score function like in a parametric ML. We can show (66) as follows. Denote $\mathcal{L}(Y,W,\alpha) = \exp(l(Y|W,\alpha))$.

$$E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha-\alpha_0]\right] = E\left[\frac{\frac{d\mathcal{L}(Y,W,\alpha_0)}{d\alpha}[\alpha-\alpha_0]}{\mathcal{L}(Y,W,\alpha_0)}\right] = \int\sum_y \frac{\frac{d\mathcal{L}(Y=y,W,\alpha_0)}{d\alpha}[\alpha-\alpha_0]}{\mathcal{L}(Y=y,W,\alpha_0)}\mathcal{L}(Y=y,W,\alpha_0)f_W(W)dW$$

$$= \lim_{\tau\to 0}\frac{d}{d\tau}\int\sum_y \mathcal{L}(Y=y,W,\alpha_0+\tau(\alpha-\alpha_0))f_W(W)dW = \lim_{\tau\to 0}\frac{d}{d\tau}\int f_W(W)dW = 0$$

where the second equality holds since $\mathcal{L}(Y=y,W,\alpha_0) = \Pr(Y=y|W,\alpha_0)$, by construction, for all $y \in \{(1,1),(1,0), (0,1),(0,0)\}$, the third equality holds by the interchangeability of integral and derivative and by definition of directional derivative, the fourth equality holds since $\sum_y \mathcal{L}(Y=y,W,\alpha_0+\tau(\alpha-\alpha_0)) = 1$ regardless of τ and α, and the last result holds since $\int f_W(W)dW = 1$.

From (66), it follows that

$$E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[v^*]\right] = 0 \text{ and} \tag{67}$$

$$E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\Pi_n v^*]\right] = 0. \tag{68}$$

We also need the following results to verify the condition (AN3).

For $\alpha_3 \in \mathcal{A}_n$ such that $\|\alpha_3 - \alpha_0\|_2 \leq \delta_n$ and for $\alpha_1,\alpha_2 \in \mathcal{A}_n - \alpha_0$, note that

$$\left|E\left[\frac{dl(Y|W,\alpha_3)}{d\alpha}[\alpha_1]\frac{dl(Y|W,\alpha_3)}{d\alpha}[\alpha_2]\right] - E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha_1]\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha_2]\right]\right|$$

$$\leq \left|E\left[\left(\frac{dl(Y|W,\alpha_3)}{d\alpha}[\alpha_1]-\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha_1]\right)\frac{dl(Y|W,\alpha_3)}{d\alpha}[\alpha_2]\right]\right| + \left|E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha_1]\left(\frac{dl(Y|W,\alpha_3)}{d\alpha}[\alpha_2]-\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha_2]\right)\right]\right|$$

$$\leq C_1\|\alpha_3-\alpha_0\|_s\|\alpha_2\|_s\|\alpha_1\|_s \leq C_1\left(\|\alpha_3-\alpha_0\|_2\|\alpha_2\|_2\|\alpha_1\|_2\right)^{\frac{2\nu_1/d_z}{2\nu_1/d_z+1}}$$

where the first inequality uses the triangle inequality, the second inequality holds by Conditions 10-12, and the last result holds by Theorem 1 of Gabushin (1967) (when ν_1/d_z is an integer) or by Lemma 2 in Chen and Shen (1998) for any $\nu_1/d_z > 0$. Thus, the condition A4 (i) in Wong and Severini (1991) holds with $\nu_1/d_z > 1$. Note also that

$$\left| E\left[\frac{dl(Y|W,\alpha_3)}{d\alpha}[\alpha_1,\alpha_2]\right] - E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\alpha_1,\alpha_2]\right]\right| = \left| E\left[\left(\frac{dl(Y|W,\alpha_3)}{d\alpha} - \frac{dl(Y|W,\alpha_0)}{d\alpha}\right)[\alpha_1,\alpha_2]\right]\right|$$

$$\leq C_1 \|\alpha_3 - \alpha_0\|_s \|\alpha_2\|_s \|\alpha_1\|_s \leq C_1 \left(\|\alpha_3 - \alpha_0\|_2 \|\alpha_2\|_2 \|\alpha_1\|_2\right)^{\frac{2\nu_1/d_z}{2\nu_1/d_z+1}} \tag{69}$$

and hence the condition A4 (ii) in Wong and Severini (1991) holds with $\nu_1/d_z > 1$. Now consider

$$K(\alpha_0,\widehat{\alpha}_n) - K(\alpha_0,\widehat{\alpha}_n \pm \varepsilon_n\Pi_n v^*)$$
$$= E\left[l(Y|W,\widehat{\alpha}_n \pm \varepsilon_n\Pi_n v^*) - l(Y|W,\widehat{\alpha}_n)\right] = E\left[\frac{dl(Y|W,\widehat{\alpha}_n \pm \widetilde{\varepsilon}_n\Pi_n v^*)}{d\alpha}[\pm\varepsilon_n\Pi_n v^*]\right]$$
$$= E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\pm\varepsilon_n\Pi_n v^*] + \frac{d^2l(Y|W,\alpha_0)}{d\alpha^2}[\widehat{\alpha}_n \pm \widetilde{\varepsilon}_n\Pi_n v^* - \alpha_0, \pm\varepsilon_n\Pi_n v^*] + o_p(n^{-1})\right]$$
$$= E\left[\frac{d^2l(Y|W,\alpha_0)}{d\alpha^2}[\widehat{\alpha}_n - \alpha_0, \pm\varepsilon_n\Pi_n v^*]\right] + o(n^{-1}) = \mp\varepsilon_n \times \langle\widehat{\alpha}_n - \alpha_0, \Pi_n v^*\rangle + o(n^{-1})$$

where the first equality holds by definition of $K(\cdot,\cdot)$, the second equality is using the mean value theorem with $\widetilde{\varepsilon}_n = o(n^{-1/2})$, the third equality is obtained using the Taylor expansion, the fourth equality is obtained using (i) $E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\pm\varepsilon_n\Pi_n v^*]\right] = \pm\varepsilon_n E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[v_n^*]\right] = 0$ by (68) and using (69) with $\widetilde{\varepsilon}_n,\varepsilon_n = o(n^{-1/2})$, the last result is from the fact that $\langle\alpha_1,\alpha_2\rangle = -E\left[\frac{d^2l(Y|W,\alpha_0)}{d\alpha^2}[\alpha_1,\alpha_2]\right]$ from (59) (see Wong and Severini (1991)). Therefore, the condition (AN3) holds since ε_n is arbitrary.

Note that the condition (AN4) (ii) immediately holds since $E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[\Pi_n v^*]\right] = 0$ by (68). Define

$$M^{(h)}(Y,W,\alpha_0) = \left\{\begin{array}{l} Y_1 Y_2 \frac{M_{11}^{(h)}(W,\alpha_0)}{\mathcal{L}_{11}} + Y_1(1-Y_2)\frac{M_{10}^{(h)}(W,\alpha_0)}{\mathcal{L}_{10}} \\ + (1-Y_1)Y_2 \frac{M_{01}^{(h)}(W,\alpha_0)}{\mathcal{L}_{01}} + (1-Y_1)(1-Y_2)\frac{M_{00}^{(h)}(W,\alpha_0)}{\mathcal{L}_{00}} \end{array}\right\}. \tag{70}$$

From this, it follows that

$$\frac{1}{n}\sum_{i=1}^n \frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\Pi_n v^* - v^*] = \frac{1}{n}\sum_{i=1}^n \frac{dl(y_i|w_i,\alpha_0)}{dh}[\Pi_n v_h^* - v_h^*]$$
$$= \frac{1}{n}\sum_{i=1}^n M^{(h)}(y_i,w_i,\alpha_0)(\Pi_n v_h^*(z_i) - v_h^*(z_i))$$

and thus the condition (AN4) (i) follows using the Chebyshev inequality from $0 = E\left[\frac{dl(Y_i|W_i,\alpha_0)}{d\alpha}[\Pi_n v^* - v^*]\right]$ by (67), (68), and from $\|\Pi_n v_h^* - v_h^*\|_\infty = o(n^{-1/4})$ by Assumption **SA-5** (iii). Now note that

$$\frac{dl(Y|W,\alpha_0)}{d\alpha}[v^*] = \frac{\partial l(Y|W,\alpha_0)}{\partial\theta'}v_\theta^* + \frac{dl(Y|W,\alpha_0)}{dh}[v_h^*]$$
$$= \left(\frac{\partial l(Y|W,\alpha_0)}{\partial\theta'} - M^{(h)}(Y,W,\alpha_0)b^*\right)(E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}\lambda$$
$$= \left(\frac{\partial l(Y|W,\alpha_0)}{\partial\theta'} - \frac{dl(Y|W,\alpha_0)}{dh}[b^*]\right)(E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}\lambda$$
$$= D_{b^*}(Y,W)(E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}\lambda$$

where the second equality is obtained using (70) and the definitions of v_θ^* and v_h^*. This implies that the condition (AN5) is satisfied by the Lindberg and Levy CLT since $E\left[\frac{dl(Y|W,\alpha_0)}{d\alpha}[v^*]\right] = 0$ and

$$E\left[\left\|\frac{dl(Y|W,\alpha_0)}{d\alpha}[v^*]\right\|^2\right]$$
$$= E\left[\lambda'(E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}D_{b^*}(Y,W)'D_{b^*}(Y,W)(E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}\lambda\right]$$
$$= \lambda'(E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}\lambda < \infty$$

by Assumption **SA-5** (ii). Now note $\sigma_{v^*}^2 \equiv \text{Var}\left(\frac{dl(Y|W,\alpha_0)}{d\alpha}[v^*]\right) = \lambda(E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}\lambda'$. Therefore, the conclusion

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} N\left(0, (E[D_{b^*}(Y,W)'D_{b^*}(Y,W)])^{-1}\right)$$

49

follows since λ is arbitrary with $\lambda \neq 0$.

Now the condition (AN2) remains to be proved. Note that the condition (AN2) is implied by

$$\sup_{\alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\| \leq \delta_n} \mu_n \left(\frac{dl(y_i|w_i, \alpha)}{d\alpha}[\Pi_n v^*] - \frac{dl(y_i|w_i, \alpha_0)}{d\alpha}[\Pi_n v^*] \right) = o_p(n^{-1/2}). \tag{71}$$

Let $\mathcal{F} = \{ \frac{dl(Y|W,\alpha)}{d\alpha}[\Pi_n v^*] : \alpha \in \mathcal{A} \}$. Condition 12 implies that $\frac{dl(y_i|w_i,\alpha)}{d\alpha}[\Pi_n v^*]$ satisfies the Lipschitz condition with respect to α and the metric $\|\cdot\|_s$. Thus, $\frac{dl(y_i|w_i,\alpha)}{d\alpha}[\Pi_n v^*]$ satisfies the condition (3.1) of Theorem 3 in Chen, Linton, and van Keilegom (2003). Note also that Θ is compact and $\mathcal{H}$ is a subset of a Hölder space. Thus, from the Lipschitz condition and the remark 3 (ii) of Chen, Linton, and van Keilegom (2003), it follows that

$$\int_0^\infty \sqrt{\log N_{[]} \left(\varepsilon, \mathcal{F}, \|\cdot\|_{L_2(P_0)} \right)} d\varepsilon < \infty \tag{72}$$

by the proof of Theorem 3 in Chen, Linton, and van Keilegom (2003). Now note

$$E \left[\left\| \frac{dl(y_i|w_i,\alpha)}{d\alpha}[\Pi_n v^*] - \frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\Pi_n v^*] \right\|_E^2 \right]$$
$$\leq C \cdot E \left[\left\| \frac{dl(y_i|w_i,\alpha)}{d\alpha}[\Pi_n v^*] - \frac{dl(y_i|w_i,\alpha_0)}{d\alpha}[\Pi_n v^*] \right\|_E \right] \tag{73}$$
$$\times \sup_{w \times y \in \mathbb{S}(W) \times \mathbb{S}(Y), \|\alpha - \alpha_0\| \leq \delta_n, \alpha \in \mathcal{A}_n} \left\| \frac{dl(y|w,\alpha)}{d\alpha}[\Pi_n v^*] - \frac{dl(y|w,\alpha_0)}{d\alpha}[\Pi_n v^*] \right\|_E \to 0 \text{ as } \|\alpha - \alpha_0\|_s \to 0$$

where the last result holds by Condition 12. Therefore, applying Lemma 1 in Chen, Linton, and van Keilegom (2003), we find that (71) is true by (72) and (73). This completes the proof.

F.4.2 Proof of Proposition 5.2

Similarly with the proof of Theorem 5.1 in Ai and Chen (2003). We can prove Proposition 5.2. We note that $\sum_{i=1}^n \left(\frac{dl(y_i|w_i,\widehat{\alpha}_n)}{d\theta_j} - \frac{dl(y_i|w_i,\widehat{\alpha}_n)}{dh}[b_j] \right)^2$ is globally convex in b_j and hence the solution of (20), $\widehat{b}_j^*$, must be bounded by $\left\| \widehat{b}_j^* \right\|_s \leq C$. Thus, we only care about the subset $\{ b \in \overline{\mathcal{B}} : \|b\|_s \leq C \}$ in the following discussion.

Note that uniformly over $b_j \in \mathcal{H}_n, \|b_j\|_s < C$, we have

$$\frac{1}{n} \sum_{i=1}^n \left(D_{b_j}(Y_i, W_i, \widehat{\alpha}_n) \right)^2 = \frac{1}{n} \sum_{i=1}^n \left(D_{b_j}(Y_i, W_i, \alpha_0) \right)^2 + o_p(1) \tag{74}$$

since $D_{b_j}(Y, W, \alpha) = D_{b_j}(Y_i, W_i, \alpha_0) + o_p(1)$ uniformly over $\|\alpha - \alpha_0\|_s = o(1)$ by Condition 12 and $\|b_j\|_s < C$. Thus, it suffices to show that $\left\| \widehat{b}_j^*(\cdot) - b_j^*(\cdot) \right\|_s = o_p(1)$ which implies

$$D_{\widehat{b}_j^*}(Y, W, \alpha_0) = D_{b_j^*}(Y_i, W_i, \alpha_0) + o_p(1). \tag{75}$$

Combining (74) and (75), we have $\frac{1}{n} \sum_{i=1}^n \left(D_{\widehat{b}_j^*}(Y_i, W_i, \widehat{\alpha}_n) \right)^2 = \frac{1}{n} \sum_{i=1}^n \left(D_{b_j^*}(Y_i, W_i, \alpha_0) \right)^2 + o_p(1)$ from which the claim follows.

Now it remains to show that $\left\| \widehat{b}_j^*(\cdot) - b_j^*(\cdot) \right\|_s = o_p(1)$ which is satisfied by Condition 12, Assumption **SA-5**, $\|\widehat{\alpha}_n - \alpha_0\|_s = o(1)$, and Lemma A.1 in Newey and Powell (2003).

G Utilizing the Mixing Distribution of Public Signals

Until now, we have ignored the possibility that we may estimate the finite mixing distribution of Z directly from the data or we may combine this with the estimation procedures we have considered in previous sections (for example, we may estimate the sieve ML estimator that maximizes (13)). Recall that Z follows a mixing distribution with the density

$$f_Z(z) = p f_{(st)}(z) + (1 - p) f_{(we)}(z).$$

Suppose $f_{(st)}(\cdot)$ and $f_{(we)}(\cdot)$ belong to some parametric family $\{f_{(\cdot)}(\cdot, \pi)\}$. Then, we can estimate p and π consistently using the EM algorithm as suggested in the literature (Everitt and Hand (1981), Titterington, Smith, and Makov (1985)). For a comprehensive treatment of mixture models, one can refer to Lindsay (1995). First, we briefly review the EM procedure tailored to our problem, following Arcidiacono and Bailey Jones (2003). Denote the unconditional likelihood of z_i as $f_Z(z; \pi, p) = p f_{(st)}(z; \pi) + (1 - p) f_{(we)}(z; \pi)$. From Bayes' theorem, $P(t_s|z_i; \pi, p)$, the probability that i is from the *strong* type conditional on z_i will be

$$P(t_s|z_i; \pi, p) = \frac{p f_{(st)}(z; \pi)}{f_Z(z; \pi, p)}. \tag{76}$$

If we maximize the sample average of the log of unconditionally-type-averaged likelihood $L(\pi, p) = \frac{1}{n} \sum_{i=1}^{n} \log(f_Z(z_i; \pi, p))$, we will obtain

$$\widehat{p} = \frac{1}{n} \sum_{i=1}^{n} P(t_s|z_i; \widehat{\pi}, \widehat{p}). \tag{77}$$

The maximum likelihood estimator $\widehat{\pi}$ must solve

$$\frac{1}{n} \sum_{i=1}^{n} \left(P(t_s|z_i; \widehat{\pi}, \widehat{p}) \frac{\partial \log f_{(st)}(z_i; \pi)}{\partial \pi} + (1 - P(t_s|z_i; \widehat{\pi}, \widehat{p})) \frac{\partial \log f_{(we)}(z_i; \pi)}{\partial \pi} \right) = 0$$

which is the first order condition of

$$\widehat{\pi} = \operatorname{argmax} \frac{1}{n} \sum_{i=1}^{n} \left(P(t_s|z_i; \widehat{\pi}, \widehat{p}) \log f_{(st)}(z_i; \pi) + (1 - P(t_s|z_i; \widehat{\pi}, \widehat{p})) \log f_{(we)}(z_i; \pi) \right). \tag{78}$$

This means that $\widehat{\pi}$ maximizes both the sample average of the log of unconditionally- and conditionally-type averaged log-likelihood. In the EM algorithm, starting from an initial value of $\pi^{(1)}$ and $p^{(1)}$, we update the conditional probability using (76) as $P(t_s|z_i; \pi^{(1)}, p^{(1)})$ in the "E" step. In the "M" step, using (77) and (78), we obtain $\pi^{(2)}$ and $p^{(2)}$. Asymptotic properties of this estimator are well-known in the literature.

However, in our problem, the mixing distribution is nonparametrically specified where $f_{(st)}(\cdot)$ and $f_{(we)}(\cdot)$ are infinite dimensional parameters. The identification of the mixture with nonparametric component functions for types has not been studied well with few exceptions (Kitamura (2004)) while Heckman and Singer (1994) provides an important study for the semiparametric models treating the mixing probability nonparametrically. Here we propose a *pseudo EM algorithm* that combines (76), (77), and the estimation procedures considered in the previous sections. The key idea is that for updating the mixing probability (like in step (77)), other than the prior mixing probability, we only need the ratio of densities for two component distributions, not individual densities.

In the first step, from (10), obtain an initial estimator of $\widehat{h}_n^{(1)}$. Using these, obtain

$$P(t_s|z; \widehat{h}_n^{(1)}) \equiv \frac{\exp(\widehat{h}_n^{(1)}(z))}{1 + \exp(\widehat{h}_n^{(1)}(z))} \text{ and } \widehat{p}_n^{(1)} = \frac{1}{n} \sum_{i=1}^{n} P(t_s|z_i; \widehat{h}_n^{(1)}) \tag{79}$$

from (5) and (24). These replace the steps (76) and (77), respectively. Now note that a directional derivative of the log likelihood function in (13) with respect to (θ, h^o) at the direction of (θ, h^o) equals to

$$\frac{1}{n} \sum_{i=1}^{n} \frac{p \exp(h^o(z_i))}{p \exp(h^o(z_i)) + (1 - p)} (\widetilde{h}^o - h^o) + \frac{1}{n} \sum_{i=1}^{n} \frac{dl(y_i|w_i, \theta, \log\left(\frac{p}{1-p}\right) + h^o(z_i))}{d(\theta, h^o)} [(\widetilde{\theta}, \widetilde{h}^o) - (\theta, h^o)]$$

and note that $\frac{p \exp(h^o(z))}{p \exp(h^o(z)) + (1-p)} = \frac{\exp(\log(p/(1-p)) + h^o(z))}{1 + \exp(\log(p/(1-p)) + h^o(z))} = \frac{\exp(h(z))}{1 + \exp(h(z))}$. Similarly with (78), this suggests that we can update θ and h^o by solving

$$\left(\widehat{\theta}_n^{(2)}, \widehat{h}_n^{o(2)} \right) \equiv \operatorname*{argmax}_{(\theta, h^o) \in \Theta \times \mathcal{H}_n} \frac{1}{n} \sum_{i=1}^{n} \frac{\exp(\widehat{h}_n^{(1)}(z_i))}{1 + \exp(\widehat{h}_n^{(1)}(z_i))} h^o(z_i) + \frac{1}{n} \sum_{i=1}^{n} l(y_i|w_i, \theta, \log\left(\frac{\widehat{p}_n^{(1)}}{1 - \widehat{p}_n^{(1)}} \right) + h^o(z_i)).$$

Let $\widehat{h}_n^{(2)}(z) = \log\left(\widehat{p}_n^{(1)}/(1 - \widehat{p}_n^{(1)}) \right) + \widehat{h}_n^{o(2)}$. Using this $\widehat{h}_n^{(2)}$, obtain $\widehat{p}_n^{(2)}$ from (79). Iterate this procedure until convergence is obtained.

H   Asymptotic Normality of p_0: Type Distribution

We may prove Proposition 5.3 by showing all the conditions in Theorem 2 of Chen, Linton, and van Keilegom (2003) hold. Most of conditions will be satisfied trivially. Here we directly prove Proposition 5.3 since it is a simple case of Chen, Linton, and van Keilegom (2003).

H.1   Proof of Proposition 5.3

We let $M(h)$ denote $\int_{\mathbb{S}(Z)} L(h(z)) f_Z(z) dz$ and $M_n(h)$ denote $\frac{1}{n} \sum_{i=1}^n L(h(Z_i))$. We have

$$\sqrt{n} \left(\widehat{p}_n - p_0 \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(L(\widehat{h}_n(Z_i)) - L(h_0(Z_i)) \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(L(h_0(Z_i)) - E(L(h_0(Z_i))) \right)$$
$$= \sqrt{n} \left(M_n(\widehat{h}_n) - M_n(h_0) \right) + \sqrt{n} \left(M_n(h_0) - M(h_0) \right) \tag{80}$$

Now let $\xi(h) = M_n(h) - M(h)$ be a stochastic process indexed by $h \in \mathcal{H}$. Then, we obtain the following stochastic equicontinuity such that for any positive sequence $\delta_n = o(1)$,

$$\sup_{\|h - h_0\|_\infty \leq \delta_n} |\xi(h) - \xi(h_0)| = o_p(n^{-1/2}) \tag{81}$$

by applying Lemma 1 of Chen, Linton, and van Keilegom (2003) after establishing that (a) $\{v(h) \equiv L(h) - M(h_0) : h \in \mathcal{H}\}$ is a Donsker class and that (b) $E\left[(v(h_1) - v(h_2))^2 \right] \to 0$ as $\|h_1 - h_2\|_\infty \to 0$ noting $E[v(h_0)] = 0$ by construction. Consider that $\{v(h) \equiv L(h) - M(h_0) : h \in \mathcal{H}\}$ is a subset of $\Lambda_{C_1}^{\nu_1}(\mathbb{S}(Z))$ and $\Lambda_{C_1}^{\nu_1}(\mathbb{S}(Z))$ is a Donsker class by Theorem 2.5.6 of van der Vaart and Wellner (1996). Thus, the condition (a) is satisfied. Now note

$$E\left[(v(h_1) - v(h_2))^2 \right] = E\left[(L(h_1) - L(h_2))^2 \right] \leq E\left[|L(h_1) - L(h_2)| \right] \sup_z |L(h_1) - L(h_2)|$$
$$= E\left[|L(h_1) - L(h_2)| \right] \sup_z \left| L'(\widetilde{h}(z)) \right| \|h_1 - h_2\|_\infty \leq \tfrac{1}{4} E\left[|L(h_1) - L(h_2)| \right] \|h_1 - h_2\|_\infty$$

since $L' = L(1 - L) \leq 1/4$ uniformly where the second equality is obtained by applying the mean value theorem and thus, the condition (b) is satisfied. Therefore, (81) holds. Now consider

$$\sqrt{n} \left(M_n(\widehat{h}_n) - M_n(h_0) \right)$$
$$= \sqrt{n} \left(M(\widehat{h}_n) - M(h_0) \right) + \sqrt{n} \left(M_n(\widehat{h}_n) - M(\widehat{h}_n) - M_n(h_0) + M(h_0) \right) = \sqrt{n} \left(M(\widehat{h}_n) - M(h_0) \right) + o_p(1) \tag{82}$$

where the last result is obtained by (81) and $\left\| \widehat{h}_n - h_0 \right\|_\infty = o_p(1)$. Now applying the mean value theorem, we have

$$\sqrt{n} \left(M(\widehat{h}_n) - M(h_0) \right) = \sqrt{n} \int_{\mathbb{S}(Z)} L'(\widetilde{h}_n(z)) \left(\widehat{h}_n(z) - h_0(z) \right) f_Z(z) dz$$
$$= \sqrt{n} \int_{\mathbb{S}(Z)} L'(h_0(z)) \left(\widehat{h}_n(z) - h_0(z) \right) f_Z(z) dz + \sqrt{n} \int_{\mathbb{S}(Z)} \left(L'(\widetilde{h}_n(z)) - L'(h_0(z)) \right) \left(\widehat{h}_n(z) - h_0(z) \right) f_Z(z) dz \tag{83}$$

where $\widetilde{h}_n$ lies between $\widehat{h}_n$ and h_0. Applying the mean value theorem again, we obtain (noting $L' = L(1 - L)$)

$$\left| L'(\widetilde{h}_n) - L'(h_0) \right| = \left| L\left(\widetilde{h}_n \right) \left(1 - L\left(\widetilde{h}_n \right) \right) - L(h_0)(1 - L(h_0)) \right|$$
$$= \left| \left(1 - L\left(\widetilde{h}_n \right) - L(h_0) \right) L'\left(\widetilde{\widetilde{h}}_n \right) \left(\widetilde{h}_n - h_0 \right) \right| \leq \tfrac{1}{4} \left| \widetilde{h}_n - h_0 \right|$$

where $\widetilde{\widetilde{h}}_n$ lies between $\widetilde{h}_n$ and h_0. It follows that

$$\sqrt{n} \int_{\mathbb{S}(Z)} \left(L'(\widetilde{h}_n(z)) - L'(h_0(z)) \right) \left(\widehat{h}_n(z) - h_0(z) \right) f_Z(z) dz \leq \frac{1}{4} \sqrt{n} \left\| \widetilde{h}_n - h_0 \right\|_\infty \left\| \widehat{h}_n - h_0 \right\|_\infty = o_p(1) \tag{84}$$

by the condition (i). Thus, we find

$$\sqrt{n} \left(M(\widehat{h}_n) - M(h_0) \right) = \sqrt{n} \int_{\mathbb{S}(Z)} L'(h_0(z)) \left(\widehat{h}_n(z) - h_0(z) \right) f_Z(z) dz + o_p(1) \tag{85}$$

from (83) and (84). From (80), (82), and (85), the claim follows by the condition (ii).

H.2 Proof of Proposition 5.4

We let $M_n^*(h) = \frac{1}{n}\sum_{i=1}^n L(h(Z_i^*))$. First, we note that the following condition holds by Giné and Zinn (1990),

$$\sup_{\|h-h_0\|_\infty \leq \delta_n} |\xi^*(h) - \xi^*(h_0)| = o_{P*}(1) \tag{86}$$

where $\xi^*(h) = M_n^*(h) - M_n(h)$. Note

$$\sqrt{n}\left(\widehat{p}_n^* - \widehat{p}_n\right) = \sqrt{n}\left(M_n^*(\widehat{h}_n^*) - M_n^*(\widehat{h}_n)\right) + \sqrt{n}\left(M_n^*(\widehat{h}_n) - M_n(\widehat{h}_n)\right) \tag{87}$$

and we have,

$$\sqrt{n}\left(M_n^*(\widehat{h}_n) - M_n(\widehat{h}_n)\right) = \sqrt{n}\left(M_n(\widehat{h}_n) - M(\widehat{h}_n)\right) + o_{P*}(1) \tag{88}$$

by Giné and Zinn (1990). Also note

$$
\begin{aligned}
&\sqrt{n}\left(M_n^*(\widehat{h}_n^*) - M_n^*(\widehat{h}_n)\right) \\
&= \sqrt{n}\left(M_n^*(\widehat{h}_n^*) - M_n(\widehat{h}_n^*) - M_n^*(h_0) + M_n(h_0)\right) - \sqrt{n}\left(M_n^*(\widehat{h}_n) - M_n(\widehat{h}_n) - M_n^*(h_0) + M_n(h_0)\right) \\
&\quad + \sqrt{n}\left(M_n(\widehat{h}_n^*) - M_n(\widehat{h}_n)\right) \\
&= \sqrt{n}\left(M_n(\widehat{h}_n^*) - M_n(\widehat{h}_n)\right) + o_{P*}(1)
\end{aligned} \tag{89}
$$

where the last equality is obtained using (86) and by the conditions (i) and (ii). Now consider

$$
\begin{aligned}
&\sqrt{n}\left(M_n(\widehat{h}_n^*) - M_n(\widehat{h}_n)\right) \\
&= \sqrt{n}\left(M(\widehat{h}_n^*) - M(\widehat{h}_n)\right) + \sqrt{n}\left(M_n(\widehat{h}_n^*) - M(\widehat{h}_n^*) - (M_n(h_0) - M(h_0))\right) \\
&\quad - \sqrt{n}\left(M_n(\widehat{h}_n) - M(\widehat{h}_n) - (M_n(h_0) - M(h_0))\right) \\
&= \sqrt{n}\left(M(\widehat{h}_n^*) - M(\widehat{h}_n)\right) + o_{P*}(1)
\end{aligned} \tag{90}
$$

by the conditions (i), (ii), and (iii). Now similarly with (85), we can show that

$$\sqrt{n}\left(M(\widehat{h}_n^*) - M(\widehat{h}_n)\right) = \sqrt{n}\int_{\mathbb{S}(Z)} L'(\widehat{h}_n)\left(\widehat{h}_n^* - \widehat{h}_n\right) dF_Z + o_{P*}(1). \tag{91}$$

Combining (87), (88), (89), (90), and (91), we note the claim follows by the condition (iv).

H.2.1 Stochastic Expansion of $\widehat{h}_n - h_0$

To check the condition (ii) in Proposition 5.3 (or the condition (iv) of Proposition 5.4), we need to derive the stochastic expansion of $\widehat{h}_n - h_0$ for a particular estimator. Here we provide a sketch of such an expansion for the sieve conditional ML estimator. Deriving the stochastic expansion of a nonparametric estimator obtained from a highly nonlinear objective function is often difficult.

To facilitate this task, we define a pseudo true value of θ and h such that

$$\alpha_{0K} \equiv (\theta_{0K}, h_{0K}) = \operatorname*{argmax}_{\theta \in \Theta, h = R^K(\cdot)'\pi \in \mathcal{H}_n} Q_K(\theta, h) \equiv E\left[l(Y_i|W_i, \theta, h(Z_i))\right] \tag{92}$$

and let $h_{0K}(\cdot) = R^K(\cdot)'\pi_{0K}$. Similarly, we let $\widehat{h}_n(\cdot) = R^{K(n)}(\cdot)'\widehat{\pi}_n$. Then,

$$\widehat{h}_n(\cdot) - h_{0K}(\cdot) = R^K(\cdot)'\left(\widehat{\pi}_n - \pi_{0K}\right) \text{ with } K = K(n). \tag{93}$$

Define

$$\Psi(h) = \int_{\mathbb{S}(Z)} L(h(z))(1 - L(h(z)))R^K(z)dF_Z(z). \tag{94}$$

Now suppose that

$$\Psi(h_0)'(\widehat{\pi}_n - \pi_{0K}) = \frac{1}{n}\sum_{i=1}^{n}\psi(\alpha_{0K}, Y_i, W_i) + o_p(n^{-1/2}), \tag{95}$$

$E[\psi(\alpha_{0K}, Y_i, W_i)] = o(1)$, $E[\|\psi(\alpha_{0K}, Y_i, W_i)\|_E^2] < \infty$. Further suppose,

$$\|h_{0K} - h_0\|_\infty = o\left(n^{-1/2}\right). \tag{96}$$

Then, we have

$$\sqrt{n}\int_{\mathbb{S}(Z)} L(h_0)(1 - L(h_0))\left(\widehat{h}_n - h_0\right)dF_Z$$
$$= \sqrt{n}\int_{\mathbb{S}(Z)} L(h_0)(1 - L(h_0))\left(\widehat{h}_n - h_{0K}\right)dF_Z + \sqrt{n}\int_{\mathbb{S}(Z)} L(h_0)(1 - L(h_0))\left(h_{0K} - h_0\right)dF_Z$$
$$= \sqrt{n}\Psi(h_0)'(\widehat{\pi}_n - \pi_{0K}) + \sqrt{n}\int_{\mathbb{S}(Z)} L(h_0)(1 - L(h_0))\left(h_{0K} - h_0\right)dF_Z = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi(\alpha_{0K}, Y_i, W_i) + o_p(1)$$

where the second equality is obtained from (93) and (94) and the third equality is obtained from (95) and (96) noting $L(\cdot)(1 - L(\cdot)) \leq 1/4$ uniformly. From this, it follows

$$V_p = \lim_{K\to\infty} E\left[(\psi(\alpha_{0K}, Y_i, W_i) + \varphi(Z_i))(\psi(\alpha_{0K}, Y_i, W_i) + \varphi(Z_i))'\right]$$

where $\varphi(Z_i) = L(h_0(Z_i)) - E[L(h_0(Z_i))]$. Therefore, to verify the condition (ii) of Proposition 5.3 holds, it suffices to show (95) and (96) are true. (95) can be shown using the first order conditions of (10) and (92). For (96), we note we often find $\|h_{0K} - h_0\|_\infty = \zeta(K)K^{-\nu_1/2d_Z}$ in the literature (Hirano, Imbens, and Ridder (2003)) where $\zeta(K) = \sup_{z\in\mathbb{S}(Z)}\|R^K(z)\|_E$.

I   Set Estimation of the Type Distribution (Proposition 6.1)

Here we derive the consistency for the set estimator of the type distribution parameter. We employ the Hausdorff metric measuring the distance between two sets whose elements are $\dim(\vartheta)$-vectors. For two such sets A and B, let the maximum distance between any points in A and B be given by $\rho(A|B) = \sup_{a\in A}\rho(a|B)$ where $\rho(a|B) = \inf\left\{\|a - b\|_E : b \in B\right\}$. Then, the Hausdorff metric is defined by $d(A, B) = \max\{\rho(A|B), \rho(B|A)\}$.

We prove Proposition 6.1 similarly with the proof of Theorem 2 in Andrews, Berry, and Jia (2004).

Proof. We let $p_n(\vartheta) = \frac{1}{n}\sum_{i=1}^{n} L(h(z_i, \delta))$ where $\vartheta = (\theta, \delta) \in \widehat{\Theta}_n$ and let $p(\vartheta) = E[L(h(Z, \delta))]$ where $\vartheta = (\theta, \delta) \in \Theta_+$. Now note that for any $\varepsilon > 0$,

$$P(d(\widehat{\mathcal{P}}_n, \mathcal{P}_+) > \varepsilon) \leq P(\rho(\widehat{\mathcal{P}}_n|\mathcal{P}_+) > \varepsilon) + P(\rho(\mathcal{P}_+|\widehat{\mathcal{P}}_n) > \varepsilon) \tag{97}$$

from the definition of $d(\cdot, \cdot)$. Let $\zeta = P(d(\widehat{\Theta}_n, \Theta_+) > \epsilon)$ for any $\epsilon > 0$ and consider

$$P(\rho(\widehat{\mathcal{P}}_n|\mathcal{P}_+) > \varepsilon) \leq P(\rho(\widehat{\mathcal{P}}_n|\mathcal{P}_+) > \varepsilon, d(\widehat{\Theta}_n, \Theta_+) \leq \epsilon) + \zeta$$
$$= P(\sup_{\vartheta\in\widehat{\Theta}_n}\inf\{|p_n(\vartheta) - p(\vartheta_+)| : \vartheta_+ \in \Theta_+\} > \varepsilon, d(\widehat{\Theta}_n, \Theta_+) \leq \epsilon) + \zeta \tag{98}$$

where the equality holds by definitions of $\rho(\cdot|\cdot)$, $\widehat{\mathcal{P}}_n$, and $\mathcal{P}_+$. Now consider that if $d(\widehat{\Theta}_n, \Theta_+) \leq \epsilon$, then $\rho(\widehat{\Theta}_n|\Theta_+) \leq \epsilon$ and for any $\vartheta \in \widehat{\Theta}_n$, there exists $\vartheta_{++} \in \Theta_+$ such that $\|\vartheta - \vartheta_{++}\|_s \leq \epsilon$. It follows that for any $\vartheta \in \widehat{\Theta}_n$ such that $\|\vartheta - \vartheta_{++}\|_s \leq \epsilon$, we have

$$p_n(\vartheta) - p(\vartheta_{++})$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(L(h(Z_i, \delta)) - L(h(Z_i, \delta_{++}))\right) + \frac{1}{n}\sum_{i=1}^{n}\left(L(h(Z_i, \delta_{++})) - E[L(h(Z_i, \delta_{++}))]\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} L(h(Z_i, \widetilde{\delta}))(1 - L(h(Z_i, \widetilde{\delta})))(h(Z_i, \delta) - h(Z_i, \delta_{++}))$$
$$+ \frac{1}{n}\sum_{i=1}^{n}\left(L(h(Z_i, \delta_{++})) - E[L(h(Z_i, \delta_{++}))]\right)$$
$$\leq \frac{1}{4}\sup_{z\in\mathbb{S}(Z)}|h(z, \delta) - h(z, \delta_{++})| + o_p(1) \leq C \cdot \epsilon \text{ for sufficiently large } n \geq \exists N, \tag{99}$$

where the second equality is obtained applying the mean value theorem ($\widetilde{\delta}$ lies between δ and δ_{++}), the first inequality holds since $L(1 - L) \leq 1/4$ uniformly and since we bound the second term of (99) by $o_p(1)$ applying

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the LLN ($\{Z_i\}_{i=1}^n$ are iid and $|L(h)| < 1$ uniformly), the last result holds applying the mean value theorem since $\sup_{\delta \in \mathcal{D}, z \in \mathbb{S}(Z)} \left\| \frac{\partial h(z,\delta)}{\partial \delta'} \right\|_E < \infty$. From this result, we have

$$\sup_{\vartheta \in \widehat{\Theta}_n} \inf \left\{ |p_n(\vartheta) - p(\vartheta_+)| : \vartheta_+ \in \Theta_+ \right\} \leq \sup_{\vartheta \in \widehat{\Theta}_n, \left\| \vartheta - \vartheta_{++} \right\|_s \leq \epsilon} |p_n(\vartheta) - p(\vartheta_{++})| \leq C \cdot \epsilon$$

for all sufficiently large $n \geq \exists N$. From this, it follows that for sufficient large n,

$$P(\sup_{\vartheta \in \widehat{\Theta}_n} \inf \left\{ |p_n(\vartheta) - p(\vartheta_+)| : \vartheta_+ \in \Theta_+ \right\} > \varepsilon, d(\widehat{\Theta}_n, \Theta_+) \leq \epsilon) \leq \varepsilon \tag{100}$$

and thus

$$P(\rho(\widehat{\mathcal{P}}_n | \mathcal{P}_+) > \varepsilon) \leq \varepsilon + P(d(\widehat{\Theta}_n, \Theta_+) > \epsilon) \tag{101}$$

from (98) and (100). An analogous argument provides

$$P(\rho(\mathcal{P}_+ | \widehat{\mathcal{P}}_n) > \varepsilon) \leq \varepsilon + P(d(\widehat{\Theta}_n, \Theta_+) > \epsilon). \tag{102}$$

Combining (97), (101), and (102), we have $P(d(\widehat{\mathcal{P}}_n, \mathcal{P}_+) > \varepsilon) \leq 2\varepsilon + 2P(d(\widehat{\Theta}_n, \Theta_+) > \epsilon)$. This proves Proposition 6.1 since $\varepsilon > 0$ is arbitrary and $d(\widehat{\Theta}_n, \Theta_+) = o_p(1)$. ∎

J   Smoothness of Conditional Probabilities

Here we note that for the conditional probabilities presented in Appendix C or E.3.2, the pathwise first and second derivatives are well-defined. This result is useful to verify Conditions 10-12 for the sieve conditional ML. It is easy to see that the pathwise derivatives are well-defined as long as $G_1(\cdot)$ and $G_2(\cdot)$ are continuously differentiable since the function $h(\cdot)$ appears only in $p(Z) = p(p, h) = \frac{\exp(\log(p/(1-p)) + h)}{1 + \exp(\log(p/(1-p)) + h)}$ and we have $\frac{dp(p,h)}{dh}[h_1 - h_2] = (1 - p(p,h))p(p,h)(h_1 - h_2)$. Therefore, for the conditional probabilities given in Appendix C or E.3.2, we have

$$\frac{dP_{ij}(Y|W,\theta,p(p,h))}{dh}[h_1 - h_2] = M_{ij}^{(h)}(h_1 - h_2), \forall i,j = 0,1,$$
$$\frac{d^2 P_{ij}(Y|W,\theta,p(p,h))}{dh^2}[h_1 - h_0, h_2 - h_0] = M_{ij}^{(h)(h)}(h_1 - h_0)(h_2 - h_0), \forall i,j = 0,1, \text{ and}$$
$$\frac{dh \partial P_{ij}(Y|W,\theta,p(p,h))}{dh \partial t}[h_1 - h_2] = M_{ij}^{(h)(t)}(h_1 - h_0), \forall i,j = 0,1 \text{ and for any element } t \text{ of } \theta,$$

where $M_{ij}^{(h)}$, $M_{ij}^{(h)(h)}$, and $M_{ij}^{(h)(t)}$ are some well-defined ordinary derivatives. The second thing to note is that those derivatives and other derivatives with respect to finite dimensional parameters are uniformly bounded by some constant since (i) $G_1(\cdot)$ and $G_2(\cdot)$ are continuously differentiable, (ii) the parameter space Θ is compact, (ii) $\mathbb{S}(W)$ is compact, (iii) $0 < p < 1$ and $0 < p(Z) < 1$, and (iv) $h(Z)$ appears only in $p(p,h)$. Therefore, the Lipschitz conditions for the conditional probabilities and the Lipschitz conditions for the pathwise first derivatives of the conditional probabilities are well satisfied. For example, in the case of the sieve conditional ML estimation, this implies that the Lipschitz conditions for the log likelihood and the Lipschitz conditions for the pathwise derivatives of the log likelihood are also well-defined.

Specific forms of derivatives for each model can be provided upon request.

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